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ON LAMBEK TORSION THEORIES, II

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In this note, generalizing recent works of Masaike [15] and Hoshino [9], we will provide another approach to the theory of QF-3 rings. We will also provide an explanation to the symmetry established by Masaike [14, Theorem 2].

Recall that a ring R is called left (resp. right) QF-3 if it has a minimal faithful left (resp. right) module, i.e., a faithful left (resp. right) module which appears as a direct summand in every faithful left (resp. right) module (see, e.g., Tachikawa [30] for details). In his recent paper [15], K. Masaike showed that a left QF-3 ring R is also right QF-3 if and only if it contains an idempotent f such that RfR is a minimal dense left ideal and every finitely solvable system of congruences $\{x \equiv fx_\lambda (\text{mod } I_\lambda)\}_{\lambda \in \Lambda}$ with each I_λ a left ideal is solvable. Generalizing this, we will provide a characterization of left and right QF-3 rings. To do so, we will introduce the notion of τ-absolutely pure rings in Section 1 and the notion of τ-semicompact modules in Section 2, where "τ-" means "relative to Lambek torsion theory". With those notions, we will show that a ring R is left and right QF-3 if and only if it is τ-absolutely pure, left and right τ-semicompact and contains idempotents e, f such that ReR and RfR are minimal dense right and left ideals, respectively.

Throughout this note, R stands for an associative ring with identity, modules are unitary modules, and torsion theories are Lambek torsion theories. Sometimes, we use the notation ${}_RX$ (resp. X_R) to stress that the module X considered is a left (resp. right) R-module. We denote by $\text{Mod}\, R$ (resp. $\text{Mod}\, R^{\text{op}}$) the category of left (resp. right) R-modules and by $(\ )^*$ both the R-dual functors. For a module X, we denote by $E(X)$ its injective envelope and by ε_X: $X \to X^{**}$ the usual evaluation map. Recall that a module X is said to be torsionless if ε_X is a monomorphism, and to be reflexive if ε_X is an isomorphism. Note that for a submodule X' of a module X, if X/X' is torsionless then $\text{Ker}\, \varepsilon_X \subset X'$. For an $X \in \text{Mod}\, R$, we denote by $\tau(X)$ its Lambek torsion submodule. Namely, $\tau(X)$ denotes a submodule of X such that $\text{Hom}_R(\tau(X), E({}_RR)) = 0$ and $X/\tau(X)$ is cogenerated by $E({}_RR)$. For also an $M \in \text{Mod}\, R^{\text{op}}$, we denote by $\tau(\text{M})$ its Lambek torsion submodule.