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CONTINUITY OF DIRECTIONAL ENTROPY

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1. Introduction

J. Milnor [3] has introduced the notion of directional entropy in cellular automata. Cellular automata can be briefly described as a dynamical system where K consists of finite alphabets and S is a continuous map on a compact space $K^{\boldsymbol{Z}^n}$ to itself which commutes with the translations of the lattice $\boldsymbol{Z}^n$. Since the space $K^{\boldsymbol{Z}^n}$ is compact, S is uniformly continuous. Hence it is not difficult to show that S is a block map (a finite code) [1]. (S is said to have a finite memory.) Since the directional entropy is defined in all directions, it can be considered as a generalization of the entropy of non cocompact subgroups. J. Milnor asked if the directional entropy is continuous.

When $n=1$, we denote by T the shift action on $K^{\boldsymbol{Z}}$. Let μ be a measure invariant under T. We assume that S preseves the measure μ. $\{T,S\}$ generate a $\boldsymbol{Z}\times\boldsymbol{N}$ action, which can be extended to a $\boldsymbol{Z}\times\boldsymbol{Z}=\boldsymbol{Z}^2$ action.

Let $(X,T,\mathscr{F},\mu)$ be an ergodic dynamical system of finite entropy. By Krieger's theorem, this system is isomorphic to a product space of finite alphabets with the shift. Without confusion, we will denote this symbolic system by $(X,T,\mathscr{F},\mu)$. Let S be a measure preserving invertable map of X generated by a block map. Hence $\{T,S\}$ generate a $\boldsymbol{Z}^2$-action on X. In this setting, Sinai [4] has shown the following: We assume $h(T^kS^l)<\infty$ for all $(k,l)\in\boldsymbol{Z}^2$. (Clearly this forces the entropy of the $\boldsymbol{Z}^2$-action to be zero). If $\{q_i/p_i\}$ converges to an irrational τ, then $\frac{1}{\sqrt{q_i^2+p_i^2}}h_{p_i,q_i}$ converges, where h_{p_i,q_i} denotes $h(T^{p^i}S^{q^i})$. Main tool of this proof is to express the directional entropy in a rational direction as an integral.(**)

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(**) In [4], he expressed the limit in the form of an integral. In private communications, we have agreed that in order to be able to write the limit as an integral, the proof needs further assumptions on T and S. (see Remark 2.1)