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POLAR SETS AS NONDEGENERATE CRITICAL SUBMANIFOLDS IN SYMMETRIC SPACES

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1. Introduction

In recent times a new approach has been taken to the study of compact symmetric spaces. This approach, begun by B-Y. Chen and T. Nagano [4], involves the study of pairs $(M^+(p), M^-(p))$ of totally geodesic submanifolds associated with closed geodesics. The submanifold $M^+(p)$, called the polar set through p, is the orbit through p of the isotropy subgroup. The space $M^-(p)$ intersects $M^+(p)$ orthogonally at p and, when M is irreducible, is usually a local product of two irreducible symmetric spaces.

The purpose of this paper is to exhibit the close connection between these pairs (together with their generalizations through the method of Borel-De Siebenthal [1]) and the Morse-Bott theory of isotropy-invariant functions on M. If $M=G/K$, we consider conjugation-invariant functions on G_1 which we pull back to (K_1-invariant functions on) M by means of the quadratic representation. In this way, we reduce our study to that of class-functions at the group level, and calculations may be restricted to a maximal torus in the group. If H is a vertex of the fundamental simplex and if $p=\exp H$ is a critical point of a class-function on G_1, then the eigenspaces of the Hessian coincide with the factoring obtained from the Borel-De Siebenthal splitting at p. Thus, the question of nondegeneracy and the calculation of the index is reduced to finding the eigenvalues corresponding to each factor. This can be done easily. To construct suitable class-functions we consider the real parts of the characters of irreducible representations. Some care must be taken with the choice of representation so that we do not obtain degenerate critical submanifolds. For example, in the case of the groups E_6 and G_2, the character of the adjoint representation has some M^+'s as degenerate critical submanifolds. The correct choice of representation usually seems to be one having lowest degree and, generically, the critical submanifolds (all of which are nondegenerate) are either M^+'s or are of the form $K_1.p$,