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A NECESSARY AND SUFFICIENT CONDITION FOR A 3-MANIFOLD TO HAVE GENUS g HEEGAARD SPLITTING (A PROOF OF HASS-THOMPSON CONJECTURE)

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1. Introduction

R.H. Bing had shown that a closed 3-manifold M is homeomorphic to S^3 if and only if every knot in M can be ambient isotoped to lie inside a 3-ball [1]. In [5], J. Hass and A. Thompson generalize this to show that M has a genus one Heegaard splitting if and only if there exists a genus one handlebody V embedded in M such that every knot in M can be ambient isotoped to lie inside V. Moreover, they conjectures that this can be naturally generalized for genus $g(>1)$. The purpose of this paper is to show that this is actually true. Namely we prove:

Main Theorem. *Let M be a closed 3-manifold. There exists a genus g handlebody V such that every knot in M can be ambient isotoped to lie inside V if and only if M has genus g Heegaard splitting.*

The proof of this goes as follows. First we generalize Myers' construction of hyperbolic knots in 3-manifolds [14] to show that, for each integer $g(\geq 1)$, every closed 3-manifold has a knot whose exterior contains no essential closed surfaces of genus less than or equal to g (Theorem 4.1). Knots with this property will be called g-characteristic knots. Then we show that, for each integer $h(\geq 1)$, there exists a knot K in M such that K cannot be ambient isotoped to a 'simple position' in any gensu h handlebody which gives a Heegaard splitting of of M. This is carried out by using good pencil argument of K. Johannson [9] (, and we note that this also can be proved by using inverse operation of type A isotopy argument of M. Ochiai [15]). By using this very complicated knot in M, we can show that if M contains a genus g handlebody as in Main Theorem, then M admits a Heegaard splitting of genus g.

This paper is organized as follows. In Section 2, we slightly generalize

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