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SOME DECOMPOSITION PROPERTIES OF INJECTIVE AND PURE-INJECTIVE MODULES

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It is well known that over a left Notherian ring any direct sum of injecive modules is again injective, and every injective module is a direct sum of indecomposable modules. Faith [6] introduced Σ-injective modules as modules M such that all direct sums of copies of M are injective. Cailleau [2] showed that a Σ-injective module is a direct sum of indecomposable modules. The concept of Σ-injective modules had several intersting developments and applications (see e.g. Faith [7]). Also, some generalizations of Σ-injective modules, such as Σ-quasi-injective modules (Cailleau-Renault [3]), or Σ-M-injecitve modules (Albu-Nastasescu [1]), were studied. Of special interest are Σ-pure-injective modules which were introduced and investigated extensively by W. Zimmermann and B. Zimmermann-Huisgen [22, 23, 24, 25]. These modules include, besides Σ-injective modules, also Π-projective modules (i.e., modules M such that all direct products of copies of M are projective).

Harada [12] studies Σ-injectivity in the context of Grothendieck categories and used it to characterize QF-categories. In this paper we continue the study of Σ-injectivity in the general categorical setting. One of our motivations comes from the fact that, by using the functor ring techniques of Gruson and Jensen [10, 11], serveral decomposition properties of (Σ-) pure-injective modules (over a ring with identity) can be obtained rather easily from the corresponding properties of injective obejcts in a Grothendieck category.

In Section 1 we will work in a Grothendieck category $\mathcal{A}$ with a family of finitely generated generators $\{G_\alpha/\alpha\in\Omega\}$. Our purpose is to study the basic properties of Σ-injective objects, but sometimes we require just some weakened forms of the injectivity. An object $M\in\mathcal{A}$ is called CS [4] (or extending [16, 17]) if every subobject of M is essential in a direct summand. Extending Okado [16], we show that if $M\in\mathcal{A}$ is a CS object such that for each $\alpha\in\Omega$, G_α has ACC on the subobjects $\{\mathrm{Ker} f/f\in\mathrm{Hom}(G_\alpha, M)\}$, then M is a direct sum of indecomposable objects (Proposition 1.5). Consequently, any CS subobject of a Σ-injective object in $\mathcal{A}$ is a direct sum of indecomposable objects (Corollary 1.6). Further, generalizing a result of Lawrence [14] on self-injective rings, we show that if A and M are objects of $\mathcal{A}$ and M is A-injective, and $\aleph$ is an in-