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ON SPECIAL VALUES AT $s=0$ OF PARTIAL ZETA-FUNCTIONS FOR REAL QUADRATIC FIELDS

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1. Introduction

1.1 Let F be a totally real algebraic number field with finite degree, $\mathfrak{a}$ a fractional ideal of F, and F_{ab} the maximal abelian extension of F. We define a map $\xi_{\mathfrak{a}}$ from the quotient space $F/\mathfrak{a}$ to the group $W(F_{ab})$ of roots of unity of F_{ab} using the deep results of Coates-Sinnott [$C-S1$], [$C-S2$] and Deligne-Ribet [$D-R$] on special values of partial zeta functions of F. Under the action of the Galois group $Gal(F_{ab}/F)$ of F_{ab} over F this map behaves formally in a manner similar to Shimura's reciprocity law for elliptic curves with complex multiplication. This reciprocity law for the map $\xi_{\mathfrak{a}}$ is also a direct consequence of those results of Coates-Sinnott and Deligne-Ribet. On the other hand we have studied in [Ar1] a certain Dirichlet series and its relationship with parital zeta functions of real quadratic fields. In particular the special values at $s=0$ of partial zeta functions of real quadratic fields essentially coincide with the residues at the pole $s=0$ of our Dirichlet series. Using those residues, we give another expression for the map $\xi_{\mathfrak{a}}$ in the case of F a real quadratic field. We also show that the expression works in a reasonable manner under the action of the Galois group $Gal(F_{ab}/F)$.

1.2 We summarize our results. For an integral ideal $\mathfrak{c}$ of a totally real algebraic number field F, denote by $H_F(\mathfrak{c})$ the narrow ray class group modulo $\mathfrak{c}$. For each integral ideal $\mathfrak{b}$ prime to $\mathfrak{c}$, we define the partial zeta-function $\zeta_{\mathfrak{c}}(\mathfrak{b}, s)$ to be the sum $\sum_{\mathfrak{a}}(N\mathfrak{a})^{-s}$, $\mathfrak{a}$ running over all integral ideals of the class of $\mathfrak{b}$ in $H_F(\mathfrak{c})$. Let $\mathfrak{a}$ be a fractional ideal of F. For each class $\bar{z}$ of the quotient space $F/\mathfrak{a}$, we take a totally positive representative element $z\in F$ of the class $\bar{z}$, and write

$$z\mathfrak{a}^{-1}=\mathfrak{f}^{-1}\mathfrak{b} \tag{1.1}$$

with coprime integral ideals $\mathfrak{f}, \mathfrak{b}$ of F. Thanks to some results of Coates-Sinnott ([C-S1], [C-S2], [Co]) and Deligne-Ribet ([D-R]), one can define a map $\xi_{\mathfrak{a}}: F/\mathfrak{a}\to W(F_{ab})$ as follows;

$$\xi_{\mathfrak{a}}(\bar{z}) = \exp(2\pi i\zeta_{\mathfrak{f}}(\mathfrak{b}, 0)), \tag{1.2}$$