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CHARACTERIZATIONS OF p-NILPOTENT GROUPS

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Introduction

Let G be a finite group and p a prime. For a p-block B of G, let $\mathrm{Irr}^0(B)$ be the set of irreducible characters of height 0 in B. Most results in this paper are related with the characters of height 0 in the principal p-block $B_0(G)$. In section 1 we shall show that G is p-nilpotent if and only if every $\chi \in \mathrm{Irr}^0(B_0(G))$ is modularly irreducible (Theorem 1.3). This result is in a sense analogous to a theorem of Okuyama and Tsushima [8]. We shall give also a characterization of p-nilpotent groups via weights [1]. In section 2 several normal subgroups associated to Ker χ, $\chi \in \mathrm{Irr}^0(B)$, are shown to be p-nilpotent. Also p-nilpotent groups are characterized via their character values (Corollary 2.10). In section 3 a question arising from a paper of Ono [9] will be discussed. Throughout this paper (K, R, k) denotes a p-modular system. We assume that K contains the $|G|$-th roots of unity. The maximal ideal of R is denoted by (π).

1. Characterizations of p-nilpotent groups

Let

$$\Lambda(G) = \{\chi;\, \chi \in \mathrm{Irr}^0(B_0(G)),\ o(\det \chi) \not\equiv 0 \pmod p\},$$

where $o(\det \chi)$ denotes the determinantal order of χ. For an irreducible Brauer character ϕ of G and a subset Λ of $\Lambda(G)$, let $\delta(\Lambda, \phi) = \sum d(\chi, \phi)\chi(1)$, where $d(\chi, \phi)$ is the decomposition number and the sum is taken over all $\chi \in \Lambda$. For brevity, put $\delta(G, \phi) = \delta(\Lambda(G), \phi)$.

The following lemma will be used frequently in the sequel.

Lemma 1.1. *If $\delta(G, \phi) \not\equiv 0 \pmod p$ for some irreducible Brauer character ϕ in $B_0(G)$ with $\phi(1) \not\equiv 0 \pmod p$, then G is p-nilpotent.*

Proof. Put $N = O^p(G)$. Since $\phi(1)$ is prime to p, $\psi := \phi_N$ is irreducible. The same is true for $\chi \in \Lambda(G)$, and the restriction gives a bijection from $\Lambda(G)$ onto the subset Ξ of G-invariant members of $\Lambda(N)$, cf. Corollary 6.28 in Isaacs [5]. From this it follows that $\delta(G, \phi) = \delta(\Xi, \psi)$. Now let Ψ be the character