Hoshiro, T.
Osaka J. Math.
30 (1993), 771–782

SOME EXAMPLES OF HYPOELLIPTIC OPERATORS OF INFINITELY DEGENERATE TYPE

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(Received June 8, 1992)

0. Introduction

The object of the present paper is to study some examples of the operators of the form

$$P = D_x^2 + a(x)D_y^2 + b(x)D_y\,, \tag{1}$$

on $\boldsymbol{R}^2$ where $D_x = -i\frac{\partial}{\partial x}$, $D_y = -i\frac{\partial}{\partial y}$, $a(x)$ and $b(x)$ are functions satisfying:

(2) (i) $a(x), b(x) \in C^\infty(\boldsymbol{R})$,
(ii) $a(x) > 0$ for $x \neq 0$, $\partial^\alpha a(0) = \partial^\alpha b(0) = 0$ for any α.

We consider here C^∞-hypoellipticity of the operator P on $x=0$. In general it is hypoelliptic if $b(x)$ is small compared with $a(x)$, and conversely, not hypoelliptic if $b(x)$ is big. Such conditions for the hypoellipticity were investigated in the previous paper [5]. But the examples considered here cannot be explained by the method of [5] (we cannot regard $b(x)$ small nor big in what follows). They are analogous to the one which A. Menikoff considered in [6], i.e., the finitely degenerate case where $a(x) = x^{2k}$ and $b(x) = bx^{k-1}$. We prove the following theorems.

Theorem 1. *Let* $a(x) = |x|^{-4}\exp(-2|x|^{-1})$ *and* $b(x) = b\cdot|x|^{-4}\exp(-|x|^{-1})$ *with b being a complex constant. Then the operator P is hypoelliptic if and only if b is not odd integer.*

Theorem 2. *Let* $a(x) = |x|^{-4}\exp(-2|x|^{-1})$ *and* $b(x) = b\cdot\operatorname{sgn} x\cdot|x|^{-4}\exp(-|x|^{-1})$ *with b being a complex constant. Then the operator P is hypoelliptic.*

REMARK 1: By the similar argument of the proof of theorem 1 in T. Morioka [8], we can conclude that P is micro-hypoelliptic when P is hypoelliptic.

The hypoellipticity of P is closely connected to the branching of singularities of solutions for the weakly hyperbolic operator $Q = -D_x^2 + a(x)D_y^2 + b(x)D_y$.