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CONVERGENCE TO A GEODESIC

Dedicated to Professor Masaru Takeuchi on his 60th birthday

NORIHITO KOISO

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0. Introduction

For a closed curve $\gamma(s)$ in a riemannian manifold M we define its energy $E(\gamma)$ by $||\dot{\gamma}||^2$. The first variation formula of E is given by $-2\langle\delta\gamma, D_{\dot{\gamma}}\dot{\gamma}\rangle$. Therefore, its Euler-Lagrange equation is $D_{\dot{\gamma}}\dot{\gamma}=0$, the equation of geodesics. We consider the corresponding parabolic equation

$$\text{(EP)} \qquad \frac{d}{dt}\gamma_t = D_{\dot{\gamma}_t}\dot{\gamma}_t\,.$$

This is locally expressed as

$$\frac{\partial}{\partial t}\gamma^i = \frac{\partial^2}{\partial s^2}\gamma^i + \Gamma_j{}^i{}_k \frac{\partial}{\partial s}\gamma^j \frac{\partial}{\partial s}\gamma^k\,,$$

which is a semi-linear heat equation.

This equation was studied by Eells and Sampson [ES], in higher dimensional case. They proved that if the manifold (M, g) is compact and has non-positive sectional curvature, then a solution γ_t exists for all time, and a *subsequence* γ_{t_i} converges to a geodesic. And it is not so difficult to show that if the manifold (M, g) has negative sectional curvature, then the solution γ_t itself converges to the geodesic.

Physically, equation (EP) represents the equation of motion of a rubber band in high viscous liquid. Therefore, it seems that the above curvature restriction is not necessary. More precisely, we have the following

Conjecture A. *If the manifold M is compact then Cauchy problem* (EP) *has a unique solution γ_t for all time.*

Conjecture B. *The solution γ_t converges to a geodesic when $t\to\infty$.*

In this paper we will show that this conjecture holds "almost always", with "a few" exceptions.

Theorem A. *If the manifold M is compact then Cauchy problem* (EP) *with C^∞ initial data has a unique solution γ_t for all time.*