Hirachi, K.
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THE SECOND VARIATION OF THE BERGMAN KERNEL OF ELLIPSOIDS

KENGO HIRACHI

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Introduction. In this paper we shall study Fefferman's asymptotic expansion of the Bergman kernel of (real) ellipsoids in C^n, $n\geq 2$. Regarding ellipsoids as perturbations of the ball, we compute the variations of the Bergmen kernel, and give the Taylor expansion of the log term coefficient to the second order in Webster's invariants. (The ellipsoids in normal form are parametrized by n real numbers, which we call Webster's invariants, and we shall consider the ellipsoids with small parameters.) As a consequence, we show that the vanishing of the log term of the Bergmen kernel characterizes the ball among these ellipsoids. In addition, we derive, from the procedure of computing the variation, a relation among the Bergmen kernels of different dimensional ellipsoids.

Let Ω be a smoothly bounded strictly pseudoconvex domain in C^n, with defining function $f>0$ in Ω. It has been known since the work of Fefferman [5] that the Bergman kernel of Ω is written in the form

$$K(z, \bar{z}) = \varphi(z)f(z)^{-n-1}+\psi(z)\log f(z)\,, \quad \text{where } \varphi, \psi\in C^\infty(\bar{\Omega})\,.$$

The coefficients φ, ψ were studied in Fefferman [6] and Graham [7], where they wrote down parts of the expansions of φ, ψ by using invariant polynomials in Moser's normal form coefficients. Fefferman expressed φ mod $O(f^{n-19})$ in terms of the Weyl invariants, and Graham in case $n=2$ determined φ explicitly. For further information on φ, in case $n\geq 3$, see [1].

In 2-dimensional case, Graham [7] further expressed the log term coefficient ψ mod $O(f^2)$ in an invariant manner, and showed that ψ vanishes if and only if the boundary is spherical, that is, locally biholomorphically equivalent to the sphere. (It is mentioned in [7] that an unpublished computation of D. Burns plays an essential role in characterizing spherical boundaries in terms of ψ.) The situation is rather simple in this 2-dimensional case, because the first invariant polynomial in ψ is linear. That polynomials is no longer linear in the higher dimensional case $n\geq 3$, and the analysis of Graham only gives its linear part. We are thus interested in getting information on the non-linear part.

We shall here compute the second variation of the log term for the class of