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Osaka J. Math.
30 (1993), 235–266

ON THE K-THEORY OF PE_7

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(Received January 30, 1992)

0. Introduction

Let E_7 be the compact, connected, simply-connected, simple Lie group of type E_7 and let PE_7 be the projective group associated with E_7. The purpose of this paper is to determine the algebras $K^*(PE_7)$ and $KO^*(PE_7)$ (Theorems 3.1 and 4.1) where K and KO denote respectively the complex and real K-theories. $K^*(PE_7)$ is already computed in [7, 9]. We study, however, it here again by the similar argument used to calculate $K^*(SO(n))$ and $KO^*(SO(n))$ in [11, 12]. Also in the same fashion we calculate $KO^*(PE_7)$ using certain results obtained in course of computation of $K^*(PE_7)$ as well as the results on $K^*(PE_7)$.

An outline of our method is as follows. Since the centre of E_7 is isomorphic to $\boldsymbol{Z}_2$, we regard E_7 as a $\boldsymbol{Z}_2$-space with the action of the centre as a subgroup. And we show that there exists a $\boldsymbol{Z}_2$-equivariant map $S^{4,0}\to E_7$, which is a homomorphism of groups, where $S^{4,0}$ is the unit quaternions S^3 with antipodal involution. This map yields a homeomorphism

$$S^{4,0}\times_{\boldsymbol{Z}_2} E_7 \approx P^3\times E_7$$

where P^3 is the real projective 3-space. Let $h=K$ or KO and let $h_{\boldsymbol{Z}_2}$ denote the $\boldsymbol{Z}_2$-equivariant h-theory. Then we have a canonical isomorphism $h^*_{\boldsymbol{Z}_2}(E_7)\cong h^*(PE_7)$ and furthermore $h^*_{\boldsymbol{Z}_2}(S^{4,0}\times E_7)\cong h^*(P^3\times E_7)$ induced by the above homeomorphism. Moreover we have a Künneth isomorphism $h^*(P^3\times E_7)\cong h^*(P^3)\otimes_{h^*(+)}h^*(E_7)$ since $h^*(E_7)$ is a free $h^*(+)$-module as mentioned below (here $+$ denotes a point). Making use of these isomorphisms and the Thom isomorphism in equivariant h-theory we carry out the calculation of $h^*(PE_7)$ by reducing to that of $h^*(P^3)\otimes_{h^*(+)}h^*(E_7)$ as in [11, 12]. For the algebras $h^*(P^3)$ and $h^*(E_7)$ we refer to [2, 5, 12] and [8, 13] respectively.

We use also the square formulas of [4, 12] (see (1.10) and (1.11) below). But we leave the 2nd exterior or exteriorlike power of the representation inserted into the functor $\beta(\ \ )$ uncalculated since it is complicated.

§1 is devoted to recalling some basic facts needed for our computation and also §2 to collecting the results on the K-groups of E_7 and P^n (for small n needed in the sequel). In §3 we compute $K^*(PE_7)$ and in §§4, 5 we determine