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THE LINEAR ISOTROPY GROUP OF $G_2/SO(4)$, THE HOPF FIBERING AND ISOPARAMETRIC HYPERSURFACES

Dedicated to Professor T. Nagano on his 60th birthday

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1. Introduction

The classification problem of isoparametric hypersurfaces in a sphere with four or six principal curvatures is still open. A hypersurface in a sphere is called isoparametric if each principal curvature is constant. When the number g of the principal curvatures is six, every principal curvature has the same multiplicity m [9], which takes value 1 or 2 [1]. In either case, the known examples belong to the family of homogeneous hypersurfaces. Recently, Dorfmeister and Neher [4] proved that an isoparametric hypersurface with $(g, m)=(6, 1)$ are homogeneous. Their argument is, however, purely algebraic, because they classify isoparametric functions rather than the hypersurfaces themselves. Unfortunately, their proof does not work for $(g,m)=(6,2)$. So it seems significant to consider the problem from a different point of view, more geometrically.

Up to now, about homogeneous hypersurfaces with $g=6$, we know merely a general fact that they are orbits of isotropy actions of certain symmetric spaces. But as a special case of the recent result [7, Proposition 3], when $h: S^7 \rightarrow S^4$ is the Hopf fibering, the inverse image $\tilde{N}=h^{-1}(N)$ of an isoparametric hypersurface N in S^4 is isoparametric with $g=2k$ where k is the number of principal curvatures of N. When $k=3$, N is known to be a tube of the Veronese surface [2] and certainly, $\tilde{N}$ is homogeneous. Since the family of homogeneous hypersurfaces in S^7 with $g=6$ is unique [13], this gives a new geometric characterization to it.

Now, it is interesting to know how the fibers S^3 of the Hopf fibering appear on $\tilde{N}$. Moreover, since $\tilde{N}$ is an orbit of the isotrpoy action of $G_2/SO(4)$ [13] and since S^7 is stratified by such orbits, it is interesting to know how this action is related with the Hopf fibration. In §2, we clarify this point in terms of a subgroup action of the linear isotropy group. In particular, concerning that $\tilde{N}$ is homeomorphic to $N \times S^3$, we show that $\tilde{N}$ is foliated by an isoparametric hypersurface with $(g, m)=(3, 1)$ which is diffeomorphic to N (Proposition 2.4).