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r-FOLD ζ-SKEW-SYMMETRIC MULTILINEAR FORMS

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Let R be a commutative ring with identity 1, and for an integer $r\geqq 2$, ζ an element of R with $\zeta^r=1$. For an R-module M and an r-fold multilinear map θ on M, we shall say that θ is ζ-skew symmetric, if $\theta(x_1, x_2, x_3, \cdots, x_r)=\zeta\,\theta(x_2, x_3, \cdots, x_r, x_1)$ holds for every elements $x_1, x_2, x_3, \cdots, x_r\in M$. In this paper, we investigate the R-module with r-fold ζ-skew symmetric multilinear map. In §1, we prove some fundamental properties on r-fold ζ-skew symmetric multilinear R-modules, which include ones on symmetric or cyclically-symmetric multilinear R-modules in [H_2] or [K_2]. In §2, we give two examples of r-fold ζ-skew-symmetric multilinear R-modules, one is the determinants of matrices, and another is a 3-fold trace form of an R-algebra. In §3, we shall show that a finitely generated ζ-skew symmetric multilinear R-module is characterized by an r-fold ζ-skew-symmetric matrix, which is an expansion of [K_1]. In §4, for a 3-fold 1-skew symmetric multilinear R-module $\langle[\boldsymbol{A}]\rangle$ defined by a 3-fold 1-skew symmetric matrix $\boldsymbol{A}$, we give some conditions for $\langle[\boldsymbol{A}]\rangle$ to be an associative R-algebra by some multiplication on $\langle[\boldsymbol{A}]\rangle$.

1. r-fold ζ-skew-symmetric multilinear R-module $(M, \theta; U)$

Let R be a commutative ring with identity 1, r a positive integer ($r\geqq 2$), ζ an element of R with $\zeta^r=1$, and U a faithful R-module.

DEFINITION For an R-module M, we shall call $(M, \theta; U)$ an r-fold *ζ-skew-symmetric multilinear R-module*, simply r-fold *ζ-skew-symmetric R-module,* if $\theta: M\times M\times\cdots\times M\to U$; $(x_1, x_2, x_3, \cdots, x_r)\rightsquigarrow\theta(x_1, x_2, x_3, \cdots, x_r)$ is an r-fold multilinear map of M into U satisfying $\theta(x_1, x_2, x_3, \cdots, x_r)=\zeta\,\theta(x_2, x_3, \cdots, x_r, x_1)$. If $\zeta=1$, r-fold 1-skew-symmetric R-module is called an r-fold cyclically symmetric R-module. By θ^* and θ_*, one denotes the following R-homomorphisms:

$$\theta_*: M\to \mathrm{Hom}_R(\otimes_R^{r-1} M, U);\ x \rightsquigarrow \theta(x, -)\,, \quad \text{and}$$
$$\theta_*: \otimes_R^{r-1} M\to \mathrm{Hom}_R(M, U);\ x_1\otimes\cdots\otimes x_{r-1} \rightsquigarrow \theta(-, x_1, \cdots, x_{r-1})\,,$$

where $\otimes_R^{r-1} M$ and $\theta(x, -)$ denote $\otimes_R^{r-1} M=M\otimes_R M\otimes_R\cdots\otimes_R M$: the tensor product of r-1-copies of M over R, and $\theta(x, -)$: $\otimes_R^{r-1} M\to U$; $x_2\otimes\cdots\otimes x_r\rightsquigarrow\theta(x, x_2, \cdots, x_r)$. $(M, \theta; U)$ is said to be *regular*, if θ^* is injective. If θ^* is in-