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ON QUASI-HOMOGENEOUS FOURFOLDS OF $SL(3)$

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Introduction

We recall that a quasi-homogeneous variety of an algebraic group G is an algebraic variety with a regular G-action which has an open dense orbit. A general theory of quasi-homogeneous varieties has been presented in Luna-Vust [5], and in particular, quasi-homogeneous varieties of $\boldsymbol{SL}(2)$ have been studied by Popov [9], Jauslin-Moser [2]. On the other hand, the geometry of smooth projective quasi-homogeneous threefolds of $\boldsymbol{SL}(2)$ has been thoroughly studied in Mukai-Umemura [7] and Nakano [8] by means of Mori theory.

In this note, we shall study and classify the smooth irreducible complete quasi-homogeneous fourfolds of $\boldsymbol{SL}(3)$. The motivation for this research comes from Mabuchi's work [6], in which the smooth complete n-folds with a non-trivial $\boldsymbol{SL}(n)$-action have been completely classified. Since $\boldsymbol{SL}(n)$-varieties of dimension less than n are obvious ones, we are interested in $\boldsymbol{SL}(n)$-varieties of dimension $n+1$. Let X be a smooth complete $\boldsymbol{SL}(n)$-variety of dimension $n+1$, and let d be the maximum of the dimensions of all orbits of X. It turns out that, if $d \leq n-1$, then $\boldsymbol{SL}(n)$-actions on X are easy, and essential problems occur when (1) $d=n+1$ (quasi-homogeneous case) and (2) $d=n$ (codimension 1 case). We hope that the investigation of the case (1) for $n=3$ in this note will be a good example toward the understanding of the structure of $\boldsymbol{SL}(n)$-varieties of dimension $n+1$.

Our main result is the classification theorem 11 of smooth complete quasi-homogeneous 4-folds of $\boldsymbol{SL}(3)$, which turns out extermely simple compared to the $\boldsymbol{SL}(2)$-case. Indeed, all the varieties appearing in the classification are rational 4-folds of very simple type.

This note is organized as follows. First in §1, we classify the closed subgroups of $\boldsymbol{SL}(3)$ of codimension 4. The author is indebted to Prof. Ariki for Proposition 1. In §2, examples of quasi-homogeneous 4-folds of $\boldsymbol{SL}(3)$ are constructed by rather ad-hok methods. Finally, in §3, the classification will be done.

In this note, algebraic varieties, algebraic groups and Lie algebras are all defined over a fixed algebraically closed field k of characteristic 0. An algebraic variety is always assumed to be reduced and irreducible, and an (algebraic)