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A CONSTRUCTION OF 3-MANIFOLDS WHOSE HOMEOMORPHISM CLASSES OF HEEGAARD SPLITTINGS HAVE POLYNOMIAL GROWTH

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1. Introduction

Let M be a closed, orientable 3-manifold. We say that $(V, V'; F)$ is a *Heegaard splitting* of M if V, V' are 3-dimensional handlebodies in M such that $M=V\cup V'$, $V\cap V'=\partial V=\partial V'=F$. Then F is called a *Heegaard surface* of M, and the genus of F is called the *genus* of the Heegaard splitting. We say that two Heegaard splittings $(V, V'; F)$ and $(W, W'; G)$ of M are *homeomorphic* if there is a self-homeomorphism f of M such that $f(F)=G$. Then we denote the number of the homeomorphism classes of the Heegaard splittings of genus g of M by $h_M(g)$ (possibly $h_M(g)=\infty$). We note that K. Johannson showed that if M is a Haken manifold, then $h_M(g)$ is finite for every g [8], [9]. In [12], F. Waldhausen asked whether $h_M(2g)=1$ provided M admits a Heegaard splitting of genus g. Casson-Gordon showed that the answer to this question is "No". In fact they showed that there exist infinitely many 3-manifolds M such that $h_M(2n)\geq 2$ for all $n\geq 3$ [5]. In this paper, we improve this result as follows.

Theorem. *For each integer $n(>1)$, there exist infinitely many Haken manifolds M such that for each integer g greater than or equal to n there exists $\binom{g-1}{n-1}$ mutually non-homeomorphic, strongly irreducible Heegaard splittings of M of genus $4n+2+2g$. In particular $h_M(4n+2+2g)>\binom{g-1}{n-1}$ for $g>n$.*

Since $\binom{g-1}{n-1}$ is a polynomial of g of degree $n-1$, we immediately have:

Corollary. *For each positive integer m, there exist infinitely many Haken manifolds M such that*

$$\limsup_{g\to\infty} \frac{h_M(g)}{g^m} = \infty .$$

Throughout this paper, we work in the piecewise linear category. All submanifolds are in general position unless otherwise specified. For the de-