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INVARIANTS OF THREE-MANIFOLDS DERIVED FROM LINKING MATRICES OF FRAMED LINKS

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Introduction. In [16], R. Kirby and P. Melvin study invariants of 3-manifolds $\tau_r (r \geq 3)$ introduced by E. Witten [38], N. Reshetikhin and V.G. Turaev [31], and W.B.R. Lickorish [25, 26, 27] (see also [18]). In particular, Kirby and Melvin calculated τ_3 and τ_4 explicitly. Let M be a closed, oriented 3-manifold obtained from an (integral) framed link L. Then $\tau_3(M)$ can be written as follows [16, §6].

$$\tau_3(M) = c^{-\sigma} \sqrt{2}^{-n} \sum_{S<L} \sqrt{-1}^{S\cdot S} .$$

Here n is the number of components of L, σ is the signature of its linking matrix, $c=\exp(\pi\sqrt{-1}/4)$, the sum is taken over all sublinks of L including the empty sublink, and $S\cdot S$ is the sum of all the entries in the linking matrix of S.

In this paper, we generalize τ_3 and define another series of invariants of 3-manifolds. Let q be a primitive N-th ($2N$-th, resp.) root of unity for an odd (even, resp.) positive integer N. Put

$$Z_N(M;q) = \left(\frac{G_N(q)}{|G_N(q)|}\right)^{-\sigma(A)} |G_N(q)|^{-n} \sum_{l\in(\boldsymbol{Z}/N\boldsymbol{Z})^n} q^{{}^t lAl} ,$$

where $G_N(q)=\sum_{h\in \boldsymbol{Z}/N\boldsymbol{Z}} q^{h^2}$ (a Gaussian sum), A is the linking matrix of L, l is regarded as a column vector, and ${}^t l$ is its transposed row vector. One can easily see that $Z_2(M;\sqrt{-1})=\tau_3(M)$. We will show that these are all invariants for M (Theorem 1.3). As Kirby and Melvin proved for $\tau_3(M)$, $Z_N(M;q)$ is also invariant under homotopy equivalence. More precisely, it is determined by the first Betti number of M and the linking pairing on Tor $H_1(M;\boldsymbol{Z})$ for any N and q (Proposition 2.5, Corollary 2.6).

We will express the absolute value of $Z_N(M;q)$ in terms of the cohomology ring of M with $\boldsymbol{Z}/N\boldsymbol{Z}$-coefficients (Theorem 3.2). When $|Z_N(M;q)|\neq 0$, we can also determine its phase (Theorem 4.5). It is a generalization of the Brown invariant $\beta(M)$ [16, §6] defined by the linking matrix using the signature and Brown's invariant [2] for $\boldsymbol{Z}/4\boldsymbol{Z}$-valued quadratic forms on a $\boldsymbol{Z}/2\boldsymbol{Z}$-vector space.

We can aslo calculate $Z_N(M;q)$ explicitly for 3-manifolds with linking pair-