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ELASTICAE IN A RIEMANNIAN SUBMANIFOLD

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For a curve $\gamma(s)$ in a riemannian manifold M we define two quantities: the length $L(\gamma)$ and the total square curvature $E(\gamma)$. A curve γ is called an elastica if it is a critical point of the functional E restricted to the space of curves of a fixed lnegth L_0. The notion of elastica is quite old. But modern approaches to it in differential geometry are rather new. J. Langer and D.A. Singer classified all closed elasticae in the euclidean space ([1]), and showed that Palais-Smale's condition (C) holds for the space of curves in a riemannian manifold ([2]).

In this paper we consider *elasticae restricted in a submanifold.* For example, let M be a compact surface of the euclidean space and C the set of all closed curves of given length in the surface. Is there a closed curve in C which minimizes the elastic energy $\bar{E}$ (defined as curves of the euclidean space)?

We will affirmatively answer to the question in a more general situation.

Theorem. *Let $\bar{M}$ be a riemannian manifold, M a compact submanifold of $\bar{M}$ and L_0 a positive real number. Let C be the space of all closed regular curves of length L_0 in M. For each $\gamma\in C$, we measure its (exterior) elastic energy $\bar{E}(\gamma)$ as a curve in the manifold $\bar{M}$. Then the infimum $\bar{E}_0$ of the energy $\bar{E}$ on the set C is attained by a C^∞ curve in C.*

Let γ be a curve in M of unit speed. We denote by $\bar{\tau}$ and τ its curvature vector as a curve of $\bar{M}$ and M, respectively. The difference of these two curvature is measured by the second fundamental form α of M in $\bar{M}$. That is,

$$\bar{\tau} = \tau+\alpha(\dot{\gamma}, \dot{\gamma}),$$
$$|\bar{\tau}|^2 = |\tau|^2+|\alpha(\dot{\gamma}, \dot{\gamma})|^2.$$

This formula leads us to a more general situation: elastic energy with "potential". That is what we treat in the following. Let M be a riemannian manifold and ϕ a C^∞-function defined on the unit tangent vector bundle over M. For a (closed) regular curve γ in M, we reparametrize it by the arc length, and define the energy density $f(\gamma)$ and the energy $F(\gamma)$ by

$$f(\gamma) = |\tau|^2+\phi(\dot{\gamma}),$$
$$F(\gamma) = \int f(\gamma)ds.$$