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Osaka J. Math.
29 (1992), 447–453

ON LAMBEK TORSION THEORIES

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(Received August 12, 1991)

According to Chase [3, Theorem 2.1], the argument of Morita [8, Theorem 1] yields that, for a left and right coherent ring, the injective envelope of the left regular module is flat if and only if this is the case with its opposite ring. In the present note, we will generalize this fact and provide conditions which are symmetrical for an arbitrary associative ring with identity.

Throughout R stands for an arbitrary associative ring with identity and all modules are unitary left or right R-modules. We denote by $\mathrm{Mod}\, R$ (resp. $\mathrm{Mod}\, R^{\mathrm{op}}$) the category of all left (resp. right) R-modules and by $(\ )^*$ both the R-dual functors. For a module X, we denote by $E(X)$ its injective envelope and by $\varepsilon_X\colon X \to X^{**}$ the usual evaluation map. For an $X \in \mathrm{Mod}\, R$, we denote by $\tau(X)$ its torsion submodule with respect to the Lambek torsion theory on $\mathrm{Mod}\, R$. Namely, $\tau(X)$ is the submodule of X such that (i) $\mathrm{Hom}_R(\tau(X), E({}_RR))=0$ and (ii) $E({}_RR)$ cogenerates $X/\tau(X)$. For also an $M \in \mathrm{Mod}\, R^{\mathrm{op}}$, we denote by $\tau(M)$ its torsion submodule with respect to the Lambek torsion theory on $\mathrm{Mod}\, R^{\mathrm{op}}$.

We will prove the following

Theorem A. *The following are equivalent.*

(a) $\tau(X)=\mathrm{Ker}\,\varepsilon_X$ *for every finitely presented* $X \in \mathrm{Mod}\, R$.

(a)$^{\mathrm{op}}$ $\tau(M)=\mathrm{Ker}\,\varepsilon_M$ *for every finitely presented* $M \in \mathrm{Mod}\, R^{\mathrm{op}}$.

(b) f^{**} *is monic for every monic* $f\colon X \to Y$ *in* $\mathrm{Mod}\, R$ *with* X *finitely generated and* Y *finitely presented.*

(b)$^{\mathrm{op}}$ g^{**} *is monic for every monic* $g\colon M \to N$ *in* $\mathrm{Mod}\, R^{\mathrm{op}}$ *with* M *finitely generated and* N *finitely presented.*

Proposition B. *Let* R *be right coherent. Then the following are equivalent.*

(a) $E({}_RR)$ *is flat.*

(b) *There is an* $E \in \mathrm{Mod}\, R$ *which is faithful, injective and flat.*

(c) $\tau(X)=\mathrm{Ker}\,\varepsilon_X$ *for every finitely presented* $X \in \mathrm{Mod}\, R$.

Proposition C. *Let* R *be right noetherian. Then the following are equivalent.*

(a) $E({}_RR)$ *is flat.*

(b) *Every finitely generated submodule of* $E(R_R)$ *is torsionless.*