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P. HALL'S STRANGE FORMULA FOR ABELIAN p-GROUPS

Dedicated to professor Tsuyoshi Ohyama's 60-th birthday

TOMOYUKI YOSHIDA

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1. Introduction

The purpose of this paper is to study some summations over non-ssomorphic abelian p-groups. In particular, for a finite group (or a group which has finitely many solutions of the equation $x^{p^n}=1$ for each $n\geq 1$) G, we study two Dirichlet series as follows:

$$S_{G^p}^{A}(z) := \sum_A{}' s(A, G)|A|^{-z},$$
$$H_{G^p}^{A}(z) := \sum_A{}' \frac{h(A, G)}{|\mathrm{Aut}\, A|}|A|^{-z},$$

where the summation is taken over a complete set of representatives of isomorphism classes of finite abelian p-groups and

$$s(A, G) := \#\{A_1\leq G \mid A_1\cong A\},$$
$$h(A, G) := |\mathrm{Hom}(A, G)|.$$

(The above series $S_{G^p}^{A}(z)$ and $H_{A^p}^{G}(z)$ are called the zeta functions of Sylow and Frobenius type in the paper [Yo91] because they appeared in the study of Sylow's third theorem and Frobenius' theorem on the number of solutions of the equation $x^n=1$ on a finite group.)

The main theorem states a relation between them:

Theorem 3.1.

$$H_{G^p}^{A}(z)/S_{G^p}^{A}(z) = \prod_{m=1}^{\infty}(1-p^{-m-z})^{-1}.$$

In particular, the left hand side is independent of G.

The proof of this theorem is based on the LDU-decomposition of the Hom-set matrix of the category of finite abelian p-groups and on the generating functions related to the Hom-set matrix. See [Yo 87], [Yo 91].