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WEAKLY-INJECTIVE RINGS AND MODULES

Dedicated to Professor Manabu Harada
on his 60th birthday

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1. Introduction

The concept of weak relative-injectivity of modules was introduced in [6] in order to study rings all of whose cyclic modules are embeddable as essential submodules of projective modules. The study of weak relative-injectivity of rings and modules relates to that of quasi-Frobenius rings, QI-rings and to rings of quotients.

An R-module M is called weakly R^n-injective if every n-element generated submodule of $E(M)$, the injective hull of M, is contained in a submodule of $E(M)$ isomorphic to M. An R-module M is called weakly-injective if it is weakly R^n-injeative for all $n>0$. The ring R is called a right weakly-injective ring if R is weakly-injective as right R-module.

Lemma 3.2 shows that weak R-injectivity is not a Morita invariant. However, if R is a weakly-injective integral domain and K is a ring Morita equivalent to R, then K is a weakly-injective ring (Theorem 3.1). Furthermore, for any nonsingular ring R, it is shown that R is a weakly-injective ring, if and only if the $n\times n$ matrix ring S over R is also a weakly-injective ring (Theorem 3.3). Among other results on the weak relative injectivity of triangular matrix rings, it is proved that if V is a $(D-D)$-space over a division ring D, then $R=\begin{pmatrix} D & V \\ 0 & D \end{pmatrix}$ is weakly R-injective if and only if $V\cong D$ (Corollary 4.6).

As an application we provide an example of an artinian nonsingular QF-3 ring R which is not weakly R-injective, answering a question raised by Professor Tachikawa during S.K. Jain's visit to Japan. Recall that a ring R is said to be right QF-3 if it has a minimal faithful right module [9]. It is well known that a nonsignular ring R is right and left QF-3 if and only if R has a two-sided semi-simple artinian complete ring of quotients and both the left socle and the right socle of R are essential in R.

2. Definitions, Notation and Preliminaries

Let M and N be right R-modules and let $E(M)$ be an injective hull of M.