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ALMOST HEREDITARY RINGS

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A. Tozaki and the author defined almost relative projective modules in [5], and the author has given a new concept of almost projective modules in [6]. On the other hand, we know that an artinian ring R is hereditary if and only if the Jacobson radical of R is projective, namely the Jacobson radical of P is projective for any finitely generated R-projective module P. Analogously we call R a right almost hereditary ring if the Jacobson radical of P is almost projective. In the first section, we shall show the following two theorems: 1) R is right almost hereditary if and only if R is a direct sum of i) hereditary rings, ii) serial (two-sided Nakayama) rings and iii) special tri-angular matrix rings over hereditary rings and serial rings in the first category; 2) R is two-sided almost hereditary if and only if R is a direct sum of hereditary rings and serial rings.

We shall give a proof of the second theorem in the third section. In the fourth section, we shall study more strong rings such that every submodule of P is again almost projective (resp. the Jacobson radidal of Q is almost projective for any finitely generated and almost projective module Q).

1. Main theorems

In this paper every ring R is an artinian ring with identity and every module M is a unitary right R-module. By $|M|$, $\mathrm{J}(M)$, $\mathrm{E}(M)$ and $\mathrm{Soc}_k(M)$ we denote *the length, the Jacobson radical, the injective hull and the* k^{th}*-lower Loewy series* of M, respetively. $\bar{M}$ means $M/\mathrm{J}(M)$. We shall denote $\mathrm{J}(R)$ by J. As is well known, if J is R-projective, then R is called a hereditary ring [1]. Analogously if J is almost projective as a right R-module [5], then we call R *a right almost hereditary ring*. We can define similarly a left almost hereditary ring. The above definition is equivalent to the following: $\mathrm{J}(P)$ is almost projective for any finitely generated projective module P. Therefore the definition of almost hereditary ring is Morita equivalent, and hence we may assume that R is a basic artinian ring, when we study the structure of R.

If R is hereditary, every submodule of P is again projective. However if R is right almost hereditary, then every submodule of P is not necessarily almost projective (see § 4).