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AN ELEMENTARY PROOF OF A THEOREM OF BREMNER

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In the paper [1] Bremner proved that the Diophantine equation

$$3x^4-4y^4-2x^2+12y^2-9=0 \tag{1}$$

has only two positive integer solutions $(x, y)=(1, 1)$ and $(3, 3)$, which was suggested by Enomoto, Ito and Noda in their research on tight 4-desings (see [2]). However, he used some of results of Cassels in biquadratic field $\boldsymbol{R}(\sqrt[4]{3})$ and the $\mathfrak{p}$-adic method of Skolem, so his proof is somewhat difficult. In 1983, Ko Chao and Sun Qi indicated that an elementary proof of Bremner's theorem would be significant (see [3]). Now such an elementary proof is given in this paper with nothing deeper than quadratic recipricity used. We describe our method as follows.

Since (1) may be reduced to $(3x^2-1)^2-3(2y^2-3)^2=1$, we have $(3x^2-1)+(2y^2-3)\sqrt{3}=u_n+v_n\sqrt{3}=(2+\sqrt{3})^n$, the latter equation denotes the general solution of the Pell's equation $U^2-3V^2=1$, n is an integer. Thus

$$2y^2=v_n+3\,. \tag{2}$$

First we assume $n=3m$. By

$$u_{3m}+v_{3m}\sqrt{3}=(u_m+v_m\sqrt{3})^3=(u_m^3+9u_m v_m^2)+(3u_m^2 v_m+3v_m^3)\sqrt{3},$$

we get

$$v_{3m}=3v_m(u_m^2+v_m^2)=3v_m(4v_m^2+1)\,,$$

so that

$$2y^2=3(4v_m^3+v_m)+3\,,$$

which leads to

$$6y_1^2=4v_m^3+v_m+1=(2v_m+1)(2v_m^2-v_m+1)\,,$$

where $y=3y_1$, $y_1>0$. Since $(2v_m+1,\ 2v_m^2-v_m+1)=1$ and $2\nmid(2v_m+1)$, $3\nmid(2v_m^2-v_m+1)$ we have