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Osaka J. Math.
28 (1991), 683–696

ON THE FACTORIZATION OF THE WHITEHEAD PRODUCT $[\iota_{2n+1}, \iota_{2n+1}]$

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(Received November 16, 1990)

1. Introduction

As is well-known, the Whitehead product $[\iota_{2n+1}, \iota_{2n+1}]$ is of order 2 if $n \neq 0$, 1 or 3. It is commonly recognized that the Whitehead product $[\iota_{2n+1}, \iota_{2n+1}]$ plays a significant role for studying the homotopy groups of spheres.

In this paper we investigate the following problem: For a given finite complex X, when can the Whitehead product $[\iota_{2n+1}, \iota_{2n+1}]$ be factorized as $\alpha\circ\beta$, where $\beta: S^{4n+1}\to X$ and $\alpha: X\to S^{2n+1}$?

Two extreme cases are known; First, when X=one point, then the classical theoiem of J.F. Adams gives the answer, that is, in this case, the above factorization happens if and only if $n=0$, 1 or 3. Second, let $\boldsymbol{F}$ be one of the fields $\boldsymbol{R}$(real), $\boldsymbol{C}$(complex) or $\boldsymbol{H}$(quaternion). Let $Q^n(\boldsymbol{F})$ be the quasi $\boldsymbol{F}$-projective space [3]. We take X as $\Sigma^{d(n+1)-1}Q^n(\boldsymbol{F})$, $d(n+1)-1$ fold suspension of the space $Q^n(\boldsymbol{F})$, where d is the dimension of $\boldsymbol{F}$ over $\boldsymbol{R}$. Let $\alpha: \Sigma^{2d(n+1)-3}\to \Sigma^{d(n+1)-1}Q^n(\boldsymbol{F})$ be the $d(n+1)-1$ fold suspension of the attaching map of the top cell of $Q^{n+1}(\boldsymbol{F})$ and $\beta: \Sigma^{d(n+1)-1}Q^n(\boldsymbol{F})\to S^{d(n+1)-1}$ be the unstable representative of the S^{d-1} transfer map. For example, we can take β as the adjoint map of the following composite;

$$Q^n(\boldsymbol{F}) \xrightarrow{r} G_{\boldsymbol{F}}(n) \xrightarrow{J} \Omega^{dn}S^{dn} \xrightarrow{\Sigma^{d-1}} \Omega^{d(n+1)-1}S^{d(n+1)-1},$$

where r is the reflection map [3], $G_{\boldsymbol{F}}(n)$ is the orthogonal $\boldsymbol{F}$-linear group, and J is the J-map. Then, for any n, $\alpha\circ\beta=[\iota_{d(n+1)-1}, \iota_{d(n+1)-1}]$. This result is due to James and Whitehead [4].

There is another known example. Let $Q^{2n+1}_{n+1}(\boldsymbol{F})$ be the stunted quasi-projective space $Q^{2n+1}(\boldsymbol{F})/Q^n(\boldsymbol{F})$. There is a canonical cofibration;

$$S^{d(n+1)-1} \to Q^{2n+1}_{n+1}(\boldsymbol{F}) \to Q^{2n+1}_{n+2}(\boldsymbol{F}) \xrightarrow{\partial} S^{d(n+1)} \to \cdots,$$

where the first map is the inclusion of the bottom sphere and the second is the pinching map of the bottom sphere. It is easy to see that there exist a complex