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MAX-FLOW MIN-CUT THEOREM IN AN ANISOTROPIC NETWORK

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1. Introduction

Max-flow problems and min-cut problems have been investigated in domains in Euclidean spaces as well as on graphs. In this paper, we shall formulate general optimization problems which contain the problems such as in [9], [15], [16] and establish max-flow min-cut theorems related to these problems.

To clarify our idea more precisely, let us begin with recalling a standard max flow min-cut theorem on networks due to Ford and Fulkerson [6]. Let α and β be two distinguished nodes of a finite connected graph G, and let c be a capacity function, that is, a nonnegative function on the set $Y=E(G)$ of arcs (= edges) in G. Let $X=V(G)$ be the set of all nodes (=vertices) in G. For each node x, denote by $Y_+(x)$ (resp. $Y_-(x)$) the set of all arcs which come from (resp. go to) x. A flow σ from α to β is a real-valued function on Y such that its net flow out of x, which is defined by

$$\sum_{y\in Y_+(x)} \sigma(y)-\sum_{y\in Y(x)} \sigma(y)\,,$$

is required to vanish for each x in X except α and β. The value of a flow σ from α to β is defined by its net flow out of α. A max-flow problem is to find the maximal flow value of σ subject to the constraint that σ is a flow from α to β and $|\sigma|\leq c$ on Y. On the other hand, a subset Q of Y is called a cut separating α and β if there exists a partition (X', X'') of X such that $\alpha\in X'$, $\beta\in X''$ and Q is the set of all arcs joining X' and X''. For a cut Q, we call the quantity $\sum_{y\in Q} c(y)$ the cut capacity. The min-cut problem related to the above max-flow problem is to find the minimal cut capacity of all cuts separating α and β. The celebrated max-flow min-cut theorem in [6] assures that the values of those problems are equal.

Now we state continuous versions of the above problems. In stead of G and $\{\alpha, \beta\}$, we take a bounded domain Ω in the n-dimensional Euclidean space R^n and mutually disjoint nonempty two subsets $\{A, B\}$ of the boundary $\partial\Omega$ of Ω. A flow σ from A to B is a vector field which satisfies the following conditions: