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INTEGRODIFFERENTIAL EQUATION WHICH INTERPOLATES THE HEAT EQUATION AND THE WAVE EQUATION (II)

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1. Introduction

In the present paper we are concerned with the integrodifferential equation

$$(\text{IDE})_\alpha \qquad u(t, x) = \phi(x) + \frac{t^{\alpha/2}}{\Gamma\left(1+\frac{\alpha}{2}\right)} \psi(x) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Delta u(s, x)\, ds \qquad t>0,\ x \in \boldsymbol{R}$$

for $1 \leq \alpha \leq 2$. Here $\Gamma(x)$ is the gamma function and $\Delta = (\partial/\partial x)^2$. When $\psi \equiv 0$, $(\text{IDE})_1$ is reduced to the heat equation. For $\alpha = 2$, $(\text{IDE})_2$ is just the wave equation and its solution $u_2(t, x)$ has the expression called the d'Alembert's formula:

$$u_2(t, x) = \frac{1}{2} [\phi(x+t) + \phi(x-t)] + \frac{1}{2} \int_{x-t}^{x+t} \psi(y)\, dy\,.$$

The present paper is the continuation of [6]; the aim of the present paper, which is different from that of [6], is to investigate the structure of the solution of $(\text{IDE})_\alpha$ by its decomposition for every α, $1 \leq \alpha \leq 2$.

In Theorem B below, we shall show that $(\text{IDE})_\alpha$ has the unique solution $u_\alpha(t, x)$ $(1 \leq \alpha \leq 2)$ expressed as

$$(1) \qquad u_\alpha(t, x) = \frac{1}{2} \boldsymbol{E}[\phi(x+Y_\alpha(t)) + \phi(x-Y_\alpha(t))] + \frac{1}{2} \boldsymbol{E} \int_{x-Y_\alpha(t)}^{x+Y_\alpha(t)} \psi(y)\, dy$$

where $Y_\alpha(t)$ is continuous, nondecreasing and nonnegative stochastic process with Mittag-Leffler distributions of order $\alpha/2$, and $\boldsymbol{E}$ stands for the expectation. We remark that the expression (1) has the same form as that of the d'Alembert's formula.

In Theorem A below, we shall consider the decomposition of $u_\alpha(t, x)$ $(1 \leq \alpha \leq 2)$. We decompose u_α into two functions u_α^+ and u_α^- defined by

$$(2) \qquad u_\alpha^+(t, x) = \frac{1}{2} \boldsymbol{E}[\phi(x-Y_\alpha(t)) - \Psi(x-Y_\alpha(t))]$$