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AN INTEGRAL REPRESENTATION ON THE PATH SPACE FOR SCATTERING LENGTH

Dedicated to Professor N. Ikeda on the occasion of his sixtieth birthday

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(Received June 22, 1989)

0. The *scattering length* Γ is the limit of the scattering amplitude $f_k(e, e')$ as the wave number k tends to 0. It is independent of the choice of unit 3-vectors e and e'. The scattering amplitude is defined as the unique constant $f_k(e, e')$ such that there holds the asymptotics

$$\phi_k(x) \sim e^{ik\langle u,e\rangle} + f_k(e, e')\, e^{ik\langle u,e'\rangle}/|x| \quad \text{as} \quad |x| \to \infty \quad \left(e' = \frac{x}{|x|}\right)$$

for a solution ϕ_k, called the *scattering solution*, of the equation

$$\Delta\phi_k - v\phi_k = -k^2 \phi_k ,$$

where v is a given potential which is assumed to be nonnegative and integrable. As M. Kac proved,

$$\Gamma = \frac{1}{2\pi} \int_{R^3} v(x)\, \phi_0(x)\, dx \tag{1}$$

where $\phi_0(x)$ is the solution of

$$\phi_0(x) = 1 - \frac{1}{2\pi} \int_{R^3} \frac{v(y)\, \phi_0(y)}{|x-y|}\, dy . \tag{2}$$

In [4], M. Kac gave the formula

$$\Gamma = \frac{1}{2\pi} \lim_{t\to\infty} \frac{1}{t} \int_{R^3} E_x[1 - \exp(-\int_0^t v(w(s))\, ds)]\, dx \tag{3}$$

where E_x denotes the expectation with respect to the three dimensional Brownian motion starting at x. He conjectured that

(C1) the scattering length $\Gamma = \Gamma(\alpha v)$ for the potential αv has limit as α goes to infinity and

(C2) the limit, say γ_U, is independent of the choice of potential v and depends only on the support $U = \{x; v(x) > 0\}$.

The purpose of the present note is to prove the conjecture C1–2 by giving an integral representation of the scattering length $\Gamma(v)$ on the path space W,