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INTEGRODIFFERENTIAL EQUATION WHICH INTERPOLATES THE HEAT EQUATION AND THE WAVE EQUATION

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Introduction

Recently many authors have studied the following integrodifferential equation:

$$u(t, x) = \phi(x)+\int_0^t h(t-s)\Delta u(s, x)ds \qquad t>0,\ x\in \boldsymbol{R} \tag{0.1}$$

where $\Delta=(\partial/\partial x)^2$. (cf. [3], [5], [7], [16], [17]). The equation (0.1) describes the heat conduction with memory ([5], [7]). In the present paper, we shall consider the case $h(t)=\frac{t^{\alpha-1}}{\Gamma(\alpha)}$ $(\equiv h_\alpha(t))$ for $1\leq\alpha\leq 2$. Here $\Gamma(x)$ is the gamma function. Thus, the equation (0.1) becomes

$$u(t, x) = \phi(x)+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Delta u(s, x)ds\,. \tag{IDE$_\alpha$}$$

For the selection of $\{h_\alpha(t)\}_{1\leq\alpha\leq 2}$, we have two reasons. The first reason is that the operator

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)ds \tag{0.2}$$

defines the Riemann-Liouville integral of order α ([11]). As a result, $(\text{IDE})_\alpha$ $(1<\alpha<2)$ interpolates the heat equatoin $(\text{IDE})_1$ and the wave equation $(\text{IDE})_2$. Formally, $(\text{IDE})_\alpha$ corresponds to "partial differential equation"

$$(\partial/\partial t)^\alpha u(t, x) = \Delta u(t, x)\,.$$

The second reason is that $\{h_\alpha(t)\}_{1<\alpha<2}$ represents memory of a long-time tail of the power order ([14]).

The aim of the present paper is to show the following for $1<\alpha<2$:

1) The fundamental solution $\frac{1}{\alpha}P_\alpha(t, |x|)$ of $(\text{IDE})_\alpha$ takes its maximum at $x=$