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BOUNDED EXPONENTIAL SUMS

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1. Introduction

T. Kamae has asked (personal communication) whether it is possible to find a sequence (a_k) of ± 1's such that the sums

$$\sum_{k=m}^{m+n} a_k e^{-ik\theta}$$

stay bounded (for all integers m and n with $n \geq 0$) for all $\theta \in [-\pi, \pi)$ (with the bound possibly depending on θ). We show that there is no such sequence. In fact, the only such real-valued sequences must be "essentially zero" in a sense explained below.

This conclusion is reached by adopting a dynamical viewpoint, applying the Spectral Theorem, and showing that every nonzero element of L^2 must have nonzero mean power at some frequency. This latter observation is equivalent to the triviality of the intersection of all the spaces of "twisted coboundaries" for a unitary operator.

2. Results

Suppose that $a=(a_k) \in \boldsymbol{R}^{\boldsymbol{Z}}$ is a doubly infinite sequence with the property that

$$|\sum_{k=m}^{m+n} a_k e^{-ik\theta}| \leq c(\theta) < \infty \qquad \text{for all } m \in \boldsymbol{Z}, \text{ all } n \geq 0, \text{ and all } \theta \in [-\pi, \pi)\,.$$

Taking $n=\theta=0$, we see that a is bounded and so takes values in a compact interval I. Let X denote the closure of the orbit of a under the shift transformation σ in the compact metric space $I^{\boldsymbol{Z}}$. Let μ be a shift-invariant Borel probability measure on X.

Given $x \in X$ and a block $B=b_0 \cdots b_n$ which appears in x, we can find a block $D=d_0 \cdots d_n$ in a such that $|b_i - d_i| < 1/(n+1)$ for $i=0, \cdots, n$. Consequently

$$|\sum_{k=0}^{n} b_k e^{-ik\theta}| \leq c(\theta)+1 \qquad \text{for all } \theta\,.$$

If $Tg = g \circ \sigma$ for $g \in L^2(X, \mu)$ and $f(x) = \pi_0 x = x_0$ for $x \in X$, we have then that