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ON DEL PEZZO FIBRATIONS OVER CURVES

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Introduction

Let $f: M \to C$ be a proper surjective holomorphic mapping of complex manifolds M, C with $\dim M = n+1 \geqq 3$, $\dim C = 1$ and let L be an f-ample line bundle on M. Such a quadruple (f, M, C, L) will be called a *Del Pezzo fibration* if (M_x, L_x) is a Del Pezzo manifold for any general point x on C, where $M_x = f^{-1}(x)$ and L_x is the restriction of L to M_x. This means $K+(n-1)L=0$ in $\mathrm{Pic}(M_x)$ for the canonical bundle K of M.

Let me explain how such a fibration appears in the classification theory of polarized manifolds. Suppose that L is an ample line bundle on a compact complex manifold M with $\dim M = m$. Then we have the following result (cf. [F7]):

Fact 1. $K+mL$ *is nef* (i.e. $(K+mL)Z \geqq 0$ for any curve Z in M) *unless* $(M, L) \simeq (\boldsymbol{P}^m, \mathcal{O}(1))$. *So* $K+tL$ *is nef for any* $t \geqq m+1$.

Fact 2. $K+(m-1)L$ *is nef unless* $(M, L) \simeq (\boldsymbol{P}^m, \mathcal{O}(1))$, *a hyperquadric in* $\boldsymbol{P}^{m+1}$ *with* $L=\mathcal{O}(1)$, $(\boldsymbol{P}^2, \mathcal{O}(2))$ *or a scroll over a smooth curve.*

For a vector bundle $\mathcal{E}$ over X, the pair $(\boldsymbol{P}(\mathcal{E}), \mathcal{O}(1))$ is called the *scroll* of $\mathcal{E}$ (or a scroll over X).

Fact 3. *Suppose that* $K+(m-1)L$ *is nef and* $m \geqq 3$. *Then* $K+(m-2)L$ *is nef except the following cases:*

a) *There is an effective divisor* E *on* M *such that* $(E, L_E) \simeq (\boldsymbol{P}^{m-1}, \mathcal{O}(1))$ *and the normal bundle of* E *is* $\mathcal{O}(-1)$.

b0) (M, L) *is a Del Pezzo manifold,* $(\boldsymbol{P}^3, \mathcal{O}(j))$ *with* $j=2$ *or* 3, $(\boldsymbol{P}^4, \mathcal{O}(2))$ *or a hyperquadric in* $\boldsymbol{P}^4$ *with* $L=\mathcal{O}(2)$.

b1) *There is a fibration* $f: M \to C$ *over curve* C *such that* (M_x, L_x) *is a hyperquadric in* $\boldsymbol{P}^m$ *with* $L=\mathcal{O}(1)$ *or* $(\boldsymbol{P}^2, \mathcal{O}(2))$ *for any general point* x *on* C.

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