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A DISCRIMINANT CRITERION FOR THE TWO DIMENSIONAL JACOBIAN PROBLEM

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1. Let $f(X, Y)$ and $g(X, Y)$ be any two polynomials with complex coefficients, i.e., $f, g \in \boldsymbol{C}[X, Y]$. The pair (f, g) is called an *automorphic pair* if there exist $u, v \in \boldsymbol{C}[X, Y]$, such that,

$$X = u(f(X, Y), g(X, Y))$$

and

$$Y = v(f(X, Y), g(X, Y))$$

The pair (f, g) is called a *Jacobian pair* if the determinant of the Jacobian matrix

$$\begin{pmatrix} \dfrac{\partial f}{\partial X} & \dfrac{\partial f}{\partial Y} \\ \dfrac{\partial g}{\partial X} & \dfrac{\partial g}{\partial Y} \end{pmatrix}$$

of (f, g) with respect to the variables X and Y, is a nonzero element of $\boldsymbol{C}$. It is easily seen that every automorphic pair is a Jacobian pair. The Jacobian problem is to determine whether every Jacobian pair is an automorphic pair or not.

2. Various equivalent formluations of this problem are known. We shall recall some of these results, relevant to our discussion, from [1].

A polynomial $f \in \boldsymbol{C}[X, Y]$ is said to have *r points at infinity,* if its homogeneous component of maximal degree (i.e., the degree form) is a product of r coprime factors. If $F(X, Y, Z)$ is the homogenization of $f(X, Y)$, and $\mathcal{E} :=\{F(X, Y, Z)=0\}$ is the curve in $\boldsymbol{P}^2$ then, $\mathcal{E}$ intersects the line at infinity, $L:=\{Z=0\}$ in precisely r distinct points. The total number of local branches of $\mathcal{E}$ at all of these r points taken together is called the *number of places of f at infinity*. Note that the number of points at infinity is not an automorphic invariant, whereas, the number of places at infinity of a nonconstant polynomial

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