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THE REAL K-GROUPS OF SO(n) FOR n≡2 MOD 4

Dedicated to Professer Shôrô Araki on his sixtieth birthday

HARUO MINAMI

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In [9], [10] we studied the algebra $KO^*(SO(n))$ for $n\equiv 0$, 1, 3 mod 4 using an idea of [7]. We first showed that a map from $P^{n-1}\times \mathrm{Spin}(n)$ to $SO(n)$ introduced in [7] to compute $K^*(SO(n))$ also induces a monomorphism in KO-theory

$$I\colon KO^*(SO(n)) \to KO^*(P^{n-1}\times \mathrm{Spin}\,(n))\,.$$

As in [7] using this embedding enabled us to compute $KO^*(SO(n))$ from $KO^*(P^{n-1}\times \mathrm{Spin}(n))$ whose structure can be obtained from the results of [1], [6], [12], [11].

The purpose of this note is to consider the remaining case, that is, $KO^*(SO(n))$ for $n\equiv 2$ mod 4. However, in the present case, the analogous homomorphism I is not a monomorphism. This must come from the fact that the simple spin representations of $\mathrm{Spin}(n)$ are neither real nor quaternionic representations. To determine the kernel and image of I so we make use of our results on the algebra structure of $KO^*(SO(n))$ for $n\equiv 1$ mod 4.

1. $KO^*(P^{n-1}\times \mathrm{Spin}(n))$

Throughout this note we regard KO and K as Z_8-graded cohomology functors using the Bott periodicity. Let $\eta_1\in KO^{-1}(+)$ and $\eta_4\in KO^{-4}(+)$ be generators of $KO^*(+)$ satisfying the relations $2\eta_1=\eta_1^3=\eta_1\eta_4=0$, $\eta_4^2=4$ and $\mu\in K^{-2}(+)$ denote the Bott class satisfying the relation $\mu^4=1$ ($+$=point).

Let c and r denote the complexification and realification homomorphisms. According to [3] we then have a useful exact sequence

$$(1.1)\qquad \cdots\to KO^{1-q}(X)\overset{\chi}{\to} KO^{-q}(X)\overset{c}{\to} K^{-q}(X)\overset{\delta}{\to} KO^{2-q}(X)\to\cdots$$

which connects KO with K where χ is multiplication by η_1 and δ is given by $\delta(\mu x)=r(x)$ for $x\in K^{2-q}(X)$.

We also assume that

$$n\equiv 2 \bmod 4 \quad\text{and}\quad a=\frac{n-2}{2}$$