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UNITARY REPRESENTATIONS AND A GENERAL VANISHING THEOREM FOR $(0, r)$-COHOMOLOGY

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1. Introduction

Let $X=G/K$ be a Hermitian symmetric space where K is a maximal compact subgroup of a connected non-compact semisimple Lie group G. We assume that G has a finite center. Let $H\subset K$ be a Cartan subgroup of G, let g, k, h be the complexifications of the Lie algebras g_0, k_0, h_0 of G, K, H, let $\Delta=\Delta(g, h)$ be the set of non-zero roots of (g, h), and let $\Delta^+\subset\Delta$ be a system of positive roots compatible with the G-invariant complex structure on X. That is, if $g_0=k_0+p_0$ is a Cartan decomposition of g_0 then the splitting of the complexified tangent space $p=p_0^{\mathcal{C}}$ of X at the origin is given by

$$p = p^+\oplus p^- \quad \text{where} \quad p^\pm = \sum_{\alpha\in\Delta_n^+} g_{\pm\alpha} \tag{1.1}$$

for g_β the root space of $\beta\in\Delta$ and $\Delta_n^+=\Delta^+\cap\Delta_n$, Δ_n=the set of noncompact roots in Δ. Let $\Delta_k^+=\Delta^+\cap\Delta_k$ where Δ_k=the set of compact roots in Δ and let $\langle Q\rangle$ be the sum of roots in Q for $Q\subset\Delta$. In particular we set $2\delta=\langle\Delta^+\rangle$ as usual, and then we can define the following subset of the dual space h^* of h: for L the character lattice of H:

$$\begin{aligned} F_0' = \{&\text{integral forms } \Lambda \text{ in } L \,|\, (\Lambda+\delta, \alpha)\neq 0 \quad \text{for each} \\ &\alpha \text{ in } \Delta \text{ and } (\Lambda+\delta, \alpha)>0 \quad \text{for each} \quad \alpha \text{ in } \Delta_k^+\}\,. \end{aligned} \tag{1.2}$$

Now let $\tau\in\hat{K}$ be an irreducible unitary representation of K with highest weight Λ relative to the positive system Δ_k^+ for (k, h). The induced homogeneous vector bundle $E_\tau=G\times_K V_\tau$ over X has a holomorphic structure (here V_τ is the representation space of τ). Let Γ be a fixed torsion free, co-compact, discrete subgroup of G. Then given $\tau\in\hat{K}$, there is a natural sheaf $\theta_\tau(\Gamma)$ over $X_\Gamma \overset{\text{def.}}{=} \Gamma\backslash X$ generated by the presheaf: $U\mapsto$abelian group of Γ-invariant holomorphic sections of E_τ on the inverse image of U under the map $X\to X_\Gamma$, where $U\subset X_\Gamma$ is an open set. The cohomology groups $H^*(X_\Gamma, \theta_\tau(\Gamma))$ of X_Γ with coefficients in

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