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CALIBRATED GEOMETRIES IN QUATERNIONIC GRASSMANNIANS

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Introduction

Let $\boldsymbol{H}$ denote the field of quaternions and $\boldsymbol{H}^n$ the set of all n-column vectors over $\boldsymbol{H}$. We regard $\boldsymbol{H}^n$ as a right $\boldsymbol{H}$-space. The object of this paper is the quaternionic Grassmannian $G_p(\boldsymbol{H}^{p+q})$, that is, the set of all right $\boldsymbol{H}$-subspaces of $\boldsymbol{H}$-dimension p in $\boldsymbol{H}^{p+q}$.

We apply the method of calibrated geometries to the invariant differential forms on the quaternionic Grassmannians and show that certain sub-Grassmannians in the quaetrnionic Grassmannians are uniquely volume minimizing in their homology classes. Strictly speaking, we prove the following theorem.

Theorem 1. *Take a right $\boldsymbol{H}$-subspace E of $\boldsymbol{H}$-dimension $p+r$ in $\boldsymbol{H}^{p+q}$. Then the sub-Grassmannian $G_p(E)$ in $G_p(\boldsymbol{H}^{p+q})$ is a volume minimizing submanifold in its real homology class. Moreover any volume minimizing submanifold in the same homology class is congruent to it.*

Here we comment on earlier results concerning Theorem 1. Gluck-Morgan-Ziller [4] proved that in the real Grassmannian $G_p(\boldsymbol{R}^{p+q})$ each sub-Grassmannian $G_p(\boldsymbol{R}^{p+r})$ for $1 \leqslant r \leqslant q-1$ is uniquely volume minimizing in its homology class if p is an even integer greater than or equal to 4. The present paper was inspired by their paper.

Berger [2] proved that the projective subplane $P^r(\boldsymbol{H})=G_1(\boldsymbol{H}^{1+r})$ in the quaternionic projective space $P^q(\boldsymbol{H})$ is volume minimizing in its homology class for $1 \leqslant r \leqslant q-1$. His method is applicable to all quaternionic Kähler manifolds and as a result of the application it follows that a compact quaternionic submanifold in a quaternionic Kähler manifold is volume minimizing in its homology class. Moreover Fomenko [3] showed that $G_1(\boldsymbol{H}^{1+r})$ in $G_p(\boldsymbol{H}^{p+q})$ is volume minimizing in its homology class for $1 \leqslant r \leqslant q-1$.

There is a homologically volume minimizing sub-Grassmannian whose underlying field is different from that of the ambient Grassmannain. $G_1(\boldsymbol{H}^k)$

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