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EXTENDING MODULES OVER COMMUTATIVE DOMAINS

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1. Introduction

A module is extending (or has the property (C_1)) if every complement submodule is a direct summand. We prove that a module over a commutative domain has this property, if and only if it is either torsion with (C_1), or the direct sum of a torsion free reduced module with (C_1) and an arbitrary injective module. The torsion case is dealt with in [6], where we also give some background and references. Here we show that a torsion free reduced module is extending if and only if it is a finite direct sum of uniform submodules, each pair of which is extending. As an application we obtain a description of all extending modules over Dedekind domains. In a subsequent paper [7] we shall discuss the extending property for direct sums of pairs of uniform modules in general.

Throughout this paper R will be a commutative domain with quotient field K. $X\subset' M$ and $Y\subset^{\oplus} M$ denote that X is an essential submodule, and Y is a direct summand, of M.

A submodule N of a module M has no proper essential extension in M, if and only if there is another submodule N' such that N is maximal with respect to $N\cap N'=0$. Such submodules N are called closed, or complements.

2. Reduction to Torsionfree Reduced Modules

Theorem 1. *Let M be a right module over an arbitrary ring R, and let $Z_2(M)$ denote its second singular submodule. Then M is extending if and only if $M=Z_2(M)\oplus N$, where $Z_2(M)$ and N are extending and $Z_2(M)$ is N-injective.*

Proof. Since $Z_2(M)$ is closed in M, by (C_1), we have $M=Z_2(M)\oplus N$, where N is non-singular. Since (C_1) is inherited by direct summands, $Z_2(M)$ and N have (C_1).

To show that $Z_2(M)$ is N-injective, let ϕ: $X\to Z_2(M)$ be a homomorphism from a submodule X of N. Consider $X':=\{x-\phi(x): x\in X\}$. By (C_1), there exists $X'\subset' X^*\subset^{\oplus} M$. Write $M=X^*\oplus Y$. Since $X'\cap Z_2(M)=0$ and since $X'\subset' X^*$, it follows that X^* is non-singular and that $Z_2(M)=Z_2(Y)$. Hence, by