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SPECTRAL PROPERTIES OF THE LAPLACE OPERATOR IN $L^p(R)$

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1. Introduction. One of the useful tools for analyzing a linear operator T in a Banach space X, if available, is a functional calculus. In general, no reasonable functional calculus may exist. If it is known that T is a closed operator then there is available a restricted functional calculus for T based on functions which are holomorphic in a neighbourhood of the spectrum $\sigma(T)$, of T, and have a limit at infinity, [4; Ch. VII]. To admit a richer functional calculus it would be expected that T should satisfy some additional properties. For $0\leq\alpha<\pi$, define the open cone $S_\alpha=\{z\in \boldsymbol{C}\setminus\{0\}; |\arg(z)|<\alpha\}$. A closed operator T in X is said to be of type ω [12], where $0\leq\omega<\pi$, if $\sigma(T)\subseteq\bar{S}_\omega$ (the bar denotes closure and, by definition, $\bar{S}_0=[0, \infty]$) and, for $0<\varepsilon<(\pi-\omega)$ there is a positive constant c_ε such that

$$||R(\lambda; T)||\leq c_\varepsilon|\lambda|^{-1}, \qquad \lambda\notin\bar{S}_{\omega+\varepsilon}.$$

Here $R(\lambda; T)$ denotes the resolvent operator of T at λ. We remark that $-T$, for the case $0\leq\omega\leq\pi/2$, is the infinitesimal generator of a holomorphic semigroup [12; Theorems 3.3.1 and 3.3.2].

In the case when X is a Hilbert space and T is of type ω there are results of A. Yagi [13] and more recently, of A. McIntosh [10], which give conditions equivalent to the existence of a functional calculus for T based on the algebra $H^\infty(S_{\omega+\varepsilon})$, for every $0<\varepsilon<(\pi-\omega)$. For example, this is the case if the purely imaginary powers T^{iu}, $u\in\boldsymbol{R}$, exist as bounded operators in X or if T satisfies certain square function estimates. However, these results are specific to Hilbert space. The situation in Banach spaces, even reflexive ones, is less clear and more complex; some positive results in this setting can be found in [2].

Perhaps one of the simplest examples to consider is the Laplace operator $L=-d^2/dx^2$ in $L^p(\boldsymbol{R})$ for $1<p<\infty$. In this case, it turns out that L is of type $\omega=0$ and, as indicated in Section 2, L has an $H^\infty(S_\varepsilon)$-functional calculus for every $\varepsilon>0$. Another algebra of functions acting on L is the space $BV(\boldsymbol{R}^+)$ of functions on $[0, \infty)$ which are of bounded variation. We note that these

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