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THE REAL K-GROUPS OF SO(n) FOR n≡3, 4 AND 5 MOD 8

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In [14] we determined the algebra structure of $KO^*(SO(n))$ for $n\equiv 0$, 1, 7 mod 8 assuming information about the K- and KO-groups of Spin(n) and P^{n-1}.

In this paper we compute $KO^*(SO(n))$ for $n\equiv 3$, 4, 5 mod 8 in the same way as in [14]. However in the present case some generators appear in dgree -5. So we first study the squares of elements in $KO^{-5}(X)$ following the method of Crabb [7] for elements in $KO^{-1}(X)$. We then provide a short exact sequence in KO_{Z_2}-theory similar to those in Lemma 4.1 of [14] which is a main tool for our computation.

We write $A\cdot g$ for an A-module with a single generator g throughout this paper.

1. Preliminaries

a) Let G be the multiplicative group consisting of ± 1. Denote by $R^{p,q}$ the $\boldsymbol{R}^{p+q}$ with non trivial G-action on the first p coordinates, and denote by $B^{p,q}$, $S^{p,q}$ and $\Sigma^{p,q}$ the unit ball and unit sphere in $R^{p,q}$ and the quotient space $B^{p,q}/S^{p,q}$ with the collapsed $S^{p,q}$ as base point respectively.

Let X be a compact G-space with base point. According to [12, 5], if $p+q\equiv 0$ mod 8 and $p\equiv 0$ mod 4, there is a Thom element $\omega_{p,q}\in\widetilde{KO}_G(\Sigma^{p,q})$, so that we have an isomorphism

$$\phi_{p,q}\colon \widetilde{KO}_G^*(X)\cong\widetilde{KO}_G^*(\Sigma^{p,q}\wedge X)$$

given by $\phi_{p,q}(x)=\omega_{p,q}\wedge x$ for $x\in\widetilde{KO}_G^*(X)$ where $\wedge$ denotes the smash product.

We now specify the elements $\omega_{8p,0}$ and $\omega_{4,4}$. Let us take $\omega_{8p,0}$ to be the element ω_p^+ given in [14; p. 793]. Then we have

$$i^*(\omega_{8p,0}) = 2^{4p-1}(1-H) \qquad in \quad \widetilde{KO}_G(\Sigma^{0,0})=RO(G) \tag{1.1}$$

where i is the inclusion of $\Sigma^{0,0}$ into $\Sigma^{8p,0}$ and $H=R^{1,0}$.

We may assume that $\psi(\omega_{8p,0})=1$ through the periodicity isomorphism, by replacing $\omega_{8p,0}$ by $-H\omega_{8p,0}$ if necessary. Here ψ denotes the forgetful homomorphism.