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THE IMBEDDING PROBLEM OF 3-MANIFOLDS INTO 4-MANIFOLDS

Dedicated to Professor Hiroshi Toda on his 60th birthday

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We consider mainly the case $n=3$ of the following general *Imbedding Problem* in the topological category:

Under what relations between an n-manifold M and an (n+1)-manifold W, both closed, connected and oriented, does there exist an imbedding from M to W?

Since the problem is trivial for $n\leq 2$, the case $n=3$ is the first appearing non-trivial case. In general, for any n, there are two kinds of imbeddings from M to W. An imbedding f from M to W is said to be of *type* I or II, according to whether $W-fM$ is connected or not. If such an imbedding f exists, then we say that *M is type* I *or* II *imbedded in W*. If f is of type II, then $W-fM$ is seen to have exactly two components, since the boundary map $\partial: H_1(W, W-fM; Z_2)\to\tilde{H}_0(W-fM; Z_2)$ is onto and there is a duality isomorphism $H_1(W, W-fM; Z_2)\cong H^n(fM; Z_2)$ $(\cong Z_2)$ (cf. Spanier [Sp; p. 342]). It is possible to characterize the type of an imbedding $f: M\to W$ in terms of homology. In fact, f is of type II or I according to whether the homomorphism $f_*: H_n(M; Z_2)\to H_n(W; Z_2)$ is trivial or not. This is proved by examining the following commutative diagram:

$$\begin{array}{ccccc} H^n(W; Z_2) & \xrightarrow{i^*} & H^n(fM; Z_2) & & \\ \cong\uparrow & & \uparrow\cong & & \\ H_1(W; Z_2) & \xrightarrow{j_*} & H_1(W, W-fM; Z_2) & \xrightarrow{\partial} & \tilde{H}_0(W-fM; Z_2) \to 0\,, \end{array}$$

where the vertical maps are the duality isomorphisms (cf. [Sp]). For example, if $\beta_1(W; Z)=0$, then we see from the Poincaré duality and the universal coefficient theorem that any imbedding from M to W is of type II. A typical example of a type I imbedding is $M\overset{\cong}{\to}1\times M\subset S^1\times M=W$. Let $n=3$. First we show that there is an estimate of $\beta_2(W; Z)$ by $\beta_1(M; Z)$ or by certain integral invariants of an infinite cyclic covering of M, provided that M is topologically type