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EINSTEIN-KÄHLER FORMS, FUTAKI INVARIANTS AND CONVEX GEOMETRY ON TORIC FANO VARIETIES

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0. Introduction

Throughout this paper, we assume that X is a nonsingular n-dimensional toric Fano variety (defined over $\boldsymbol{C}$), i.e., X is an n-dimensional connected projective algebraic manifold satisfying the following conditions:

(a) X admits an effective almost homogeneous algebraic group action of $(\boldsymbol{G}_m)^n$ ($\cong(\boldsymbol{C}^*)^n$ as a complex Lie group).
(b) The set $\mathcal{K}$ of all Kähler forms on X in the de Rham cohomology class $2\pi\, c_1(X)_{\boldsymbol{R}}$ is non-empty.

For each $\omega\in\mathcal{K}$, by writing it as $\omega=\sqrt{-1}\sum g(\omega)_{\alpha\bar\beta}\, dz^\alpha\wedge d\bar z^\beta$ in terms of holomorphic local coordinates $(z^1, z^2, \cdots, z^n)$ of X, we have the corresponding Ricci form $\mathrm{Ric}(\omega)$ cohomologous to ω:

$$\mathrm{Ric}(\omega):=\sqrt{-1}\,\bar\partial\partial\log\det(g(\omega)_{\alpha\bar\beta})\,.$$

Then an element ω of $\mathcal{K}$ is called an Einstein-Kähler form if $\mathrm{Ric}(\omega)=\omega$. We now pose the following:

Problem 0.1*⁾. *Classify all X which admit, at least, one Einstein-Kähler form.*

Obviously, the Fubini-Study form on $\boldsymbol{P}^n(\boldsymbol{C})$ is a typical Einstein-Kähler form. This settles Problem 0.1 for $n=1$, because the only possible X with $n=1$ is $\boldsymbol{P}^1(\boldsymbol{C})$. However, the real difficulty comes up even at $n=2$: Let S_i be the projective algebraic surface obtained from $\boldsymbol{P}^2(\boldsymbol{C})$ by blowing up i points in general position (where $1\leqq i\leqq 3$). Then, in spite of lots of efforts by differential geometers, it is still unknown whether or not the nonsingular toric Fano variety S_3 admits an Einstein-Kähler form.

The purpose of this paper is to give a brief survey of recent progress on Problem 0.1 together with our related new results. Especially, in Sections 1∼6

*⁾ This is also posed by T. Oda and Y.T. Siu.