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SOME SYMPLECTIC GEOMETRY ON COMPACT KÄHLER MANIFOLDS (I)

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0. Introduction

In real Riemannian geometry, the space of all Riemannian metrics of a given compact differentiable manifold admits a Riemannian structure*) to provide us with several nice theories. In this paper, we shall seek its complex analogue. Namely, in view of the fact that all Kähler manifolds are symplectic, we shall define a very natural Riemannian structure (slightly different from classical ones) on the space of all Kähler metrics in a fixed cohomology class of a given compact Kähler manifold (see also [10] for more algebraic geometric treatments).

Throughout this paper, we fix an n-dimensional compact complex connected manifold X with a cohomology class $h \in H^{1,1}(X)_{\mathbf{R}}$ such that

$$\mathcal{K} := \{\omega \mid \omega \text{ is a Kähler form on } X \text{ in the class } h\}$$

is nonempty. Let $\omega_0 \in \mathcal{K}$ and consider the $\mathcal{K}$-energy map $\mu \colon \mathcal{K} \to \mathbf{R}$ of the Kähler manifold (X, ω_0) introduced in [9]. Now the main purpose of this paper is to define a natural Riemannian structure on $\mathcal{K}$ such that

(0.1) *μ is a convex function on $\mathcal{K}$*, i.e., Hess *μ is positive semidefinite everywhere on $\mathcal{K}$* (cf. §5);

(0.2) *sectional curvature of $\mathcal{K}$ is explicitly written in terms of Poisson brackets of functions and moreover it is always nonpositive* (cf. §4).

We next assume that

$$\mathcal{E} := \{\omega \in \mathcal{K} \mid \omega \text{ has a constant scalar curvature}\}$$

is nonempty. Recall that the Albanese map $\alpha \colon X \to \mathrm{Alb}(X)$ of X naturally induces the Lie group homomorphism $\tilde{\alpha} \colon \mathrm{Aut}^0(X) \to \mathrm{Aut}^0(\mathrm{Alb}(X))$ $(\cong \mathrm{Alb}(X))$, where $\mathrm{Aut}^0(X)$ (resp. $\mathrm{Aut}^0(\mathrm{Alb}(X))$) denotes the identity component of the

*) See, for instance, Ebin's article "The manifold of Riemannian metrics" in Global Analysis (Proc. Symp. Pure Math.) 15 (1968), 11–40.