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Osaka J. Math.
24 (1987), 13-35

HYPOELLIPTICITY FOR INFINITELY DEGENERATE ELLIPTIC OPERATORS

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(Received September 18, 1985)

Introduction. In the recent paper [5] Kusuoka-Strook gave a sufficient condition of hypoellipticity for degenerate elliptic operators of second order, as an application of the Malliavin calculus (see Theorem 8.13 of [5], cf. [4]). Their method is applicable even to infinitely degenerate elliptic operators which do not satisfy the famous sufficient condition given by Hörmander [2]. One of remarkable results by means of their condition is as follows: Let L be a differential operator of the form $\partial_{x_1}^2+\partial_{x_2}^2+\sigma(x_1)^2\partial_y^2$ in R^3, where $\sigma\in C^\infty$, $\sigma(0)=0$, $\sigma(x_1)>0\ (x_1\neq 0)$, $\sigma(x_1)=\sigma(-x_1)$ and σ is non-decreasing in $[0, \infty)$. Then L is hypoelliptic in R^3 if σ satisfies

$$(*)\qquad \lim_{x_1\downarrow 0}|x_1 \log \sigma(x_1)|=0 \qquad \text{(Theorem 8.41 of [5]).}$$

The condition (*) allows the infinite degeneracy of σ at $x_1=0$. For example, if $\sigma(x_1)=\exp(-1/|x_1|^\delta)$ for $\delta>0$ the condition (*) means $\delta<1$. The main purpose of the present paper is to show the sufficiency of the condition (*) by using the theory of pseudodifferential operators. In [5] it is proved that the condition (*) is necessary for L to be hypoelliptic. The author [7] has given a simple proof of the necessity of (*) without using the Malliavin calculus. The arguments in [7] apply to degenerate elliptic operators of higher order (see Theorem 3 of [7]).

As to the operator L we remark that an operator $\partial_{x_1}^2+\sigma(x_1)^2\partial_y^2\ (=L-\partial_{x_2}^2)$ is hypoelliptic in $R^2_{x_1,y}$ without the condition (*). This result is due to Fediǐ [1] (cf. [6]), who studied the criterion of hypoellipticity by means of apriori estimates. Such criteria have been investigated by Treves [9] and Oleinik-Radkevich [8]. Our proof of the hypoellipticity of L will be done by improving criteria studied by [8] and [1].

To explain the idea of the present paper we consider a simple case $\sigma(x_1)=\exp(-1/|x_1|^\delta)$, $\delta>0$. Then L degenerates infinitely at $x_1=0$, and hence Hörmander's sufficient condition does not apply to L. In the proof of hypoellipticity by means of apriori estimates, the technical difficulty comes from the fact that for any $\kappa>0$ subelliptic estimate