Morisugi, K.
Osaka J. Math.
23 (1986), 867–880

HOMOTOPY GROUPS OF SYMPLECTIC GROUPS AND THE QUATERNIONIC JAMES NUMBERS

KAORU MORISUGI

(Received September 13, 1985)

0. Introduction

Let $Sp(m)$ $(0 \leqq m \leqq \infty)$ be the m-th symplectic group. For convenience we denote $Sp(\infty)$ by Sp. Our purpose is to determine the homotopy groups of $Sp(m)$, $\pi_i(Sp(m))$. If $i<4m+2$ then $\pi_i(Sp(m))$ is isomorphic to $\pi_i(Sp)$. And $\pi_i(Sp)$ is well-known by Bott periodicity. Suppose $i \geqq 4m+2$, then it is not difficult to see that if $i=0$, 1, 3 or 7 mod 8 then $\pi_i(Sp(m))$ is isomorphic to $\pi_{i+1}(Sp/Sp(m))$ and if $i=4$ or 5 mod 8 then $\pi_i(Sp(m))$ is isomorphic to $\pi_{i+1}(Sp/Sp(m))+Z/2$, (direct sum), where $Sp/Sp(m)$ is the factor space of Sp by the subgroup $Sp(m)$. Thus if $i \neq 2$ mod 4 then the calculation of π_i $(Sp(m))$ can be reduced to that of $\pi_{i+1}(Sp/Sp(m))$. In the meta-stable range of i, $4m+2 \leqq i \leqq 8m+4$, $\pi_i(Sp/Sp(m))$ is isomorphic to $\pi_i^s(Q_{n,n-m})$ for sufficiently large n, where $Q_{n,n-m}$ is the stunted quaternionic quasi-projective space [8]. And when the value of i-$4m$ is small we can calculate the group $\pi_i^s(Q_{n,n-m})$ (see [15]). On the other hand in the case that $i=2$ mod 4, even if we know the group $\pi_{i+1}(Sp/Sp(m))$ this is not sufficient to determine $\pi_i(Sp(m))$. Let $i=4n-2$. There are two steps in the computation of $\pi_{4n-2}(Sp(m))$; one is to determine the quaternionic James number and the other to solve a certain group extension problem. Let us explain these. Let $X_{n,k}$ be the quaternionic Stiefel manifold of all symplectic k-frames in H^n (n-dimensional vector space over the quaternions H) and let $p\colon X_{n,k} \to S^{4n-1}$ be the bundle projection which associates with each frame its last vector. Then the quaternionic James number $X\{n, k\}$ is defined as the index of $p_*\pi_{4n-1}(X_{n,k})$. Thus $X\{n, 1\}=1$, $X\{n, l\}$ divides $X\{n, k\}$ if $l<k$ and, by the classical work of Bott [3], $X\{n, n\}=a(n-1)\cdot(2n-1)!$, where $a(\mathrm{i})=2$ if i is odd and $=1$ if i is even. It is well-known that $X_{n,k}$ is homeomorphic to $Sp(n)/Sp(n-k)$. Let $d(n, m)=X\{n, n\}/X\{n, n-m\}$. Then there exists a short exact sequence (*):

$$0 \to \mathrm{Tor}(\pi_{4n-1}(X_{n,n-m})) \xrightarrow{\Delta} \pi_{4n-2}(Sp(m)) \xrightarrow{i_*} Z/d(n, m) \to 0\,,$$

where Δ is the restriction to $\mathrm{Tor}(\pi_{4n-1}(X_{n,n-m}))$ (the torsion subgroup of π_{4n-1} $(X_{n,n-m})$) of the boundary homomorphism $\Delta'\colon \pi_{4n-1}(X_{n,n-m}) \to \pi_{4n-2}(Sp(m))$ asso-