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ON EXTREMAL QUASICONFORMAL MAPPINGS WITH VARYING DILATATION BOUNDS

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1. Introduction

Let a homeomorphism of the boundary of the unit disk $D=\{w \mid |w|<1\}$ onto itself be given which can be extended to a quasiconformal mapping of the whole disk onto itself. Then one asks for extremal extensions, i.e. quasiconformal extensions with the smallest possible maximal dilatation. This problem has been posed by Teichmüller ([21], p. 184) and main contributions have been made by Strebel ([17], [18]). Later Hamilton [7] and Reich-Strebel [14] have derived necessary and sufficient conditions for extremality of a mapping, the Hamilton-condition.

Due to a remark of Teichmüller ([21], p. 15, lines 16–20) this problem had been generalized first by Kühnau [9] in certain cases and then by Reich [12] in the following way. In addition to the fixed given boundary values the competing quasiconformal mappings are supposed to have their dilatations pointwise bounded by a prescribed function on a certain given subset E of D. Then their maximal dilatations in the remaining part of the disk have to be minimized.

In the analogous way as the Hamilton-condition is derived in the former case $E=\emptyset$, Reich has treated this problem in [12] and for a complete description of the solution we refer to section 2. Since our paper contains direct developments of Reich's work, we hope readers to be well acquainted with his paper. Mainly our contributions deal with the question how extremal mappings and certain quadratic differentials related to them depend on the dilatation bounds when these vary. To make this clear let us now explain the situation precisely.

Given are a quasisymmetric boundary mapping h: $\partial D \rightarrow \partial D$ and a closed set σ on ∂D, which is always assumed to contain at least four points. In addition a (possibly empty) measurable subset E of D such that $D \setminus E$ has positive measure is given and the dilatation bound b on E. The latter is a measurable function $b(w) \geqq 0$ on E with $\|b\|_\infty := \operatorname{ess\,sup}_{w \in E} b(w) < 1$. We consider the set $Q = Q(h, \sigma, E, b)$ of all qc (quasiconformal) mappings F: $D \rightarrow D$ with $F|_\sigma = h|_\sigma$ and with complex dilatation κ_F that satisfies