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# $p$-RADICAL GROUPS ARE $p$-SOLVABLE

Dedicated to Professor Hirosi Nagao for his 60th birthday

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Let $k$ be an algebraically closed field of characteristic $p>0$. Let $G$ be a finite group with Sylow $p$-subgroup $P$. Following Motose and Ninomiya [3] we call $G$ $p$-radical if $k_P{}^G$ is completely reducible, where $k_P$ is the trivial $kP$-module. Our aim in this paper is to prove the following theorems.

**Theorem 1.** *If $G$ is $p$-radical, then $G$ is $p$-solvable.*

**Theorem 2.** *Let $G$ be a $p$-radical group with Sylow $p$-subgroup $P$. Then the following hold;*

(1) *If $D=P\cap P^x$ for some $x$ in $G$, then $D$ is a vertex of some simple $kG$-module.*

(2) *If $D=P\cap P^x$ for some $x$ in $C_G(D)$, then $D$ is a defect group of some $p$-block of $G$.*

We will write $V\mid W$ if a $kG$-module $V$ is isomorphic to a direct summand of a $kG$-module $W$. For $kG$-modules $V$ and $W$ and a subgroup $H$ of $G$, let $(V, W)^G=\mathrm{Hom}_{kG}(V, W)$ and $(V, W)^G_H=T_{H,G}(V, W)^H$, where $T_{H,G}$ is the trace map from $(V, W)^H$ to $(V, W)^G$.

## 1. Preliminaries

Throughout this paper we let $G$ be a $p$-radical group with Sylow $p$-subgroup $P$ and put $Y=k_P{}^G$. In this section we shall prove two lemmas which will be used to prove the theorems stated in the introduction.

**Lemma 1.** *If $S$ is a simple $kG$-module with vertex $Q$, then every indecomposable direct summand of $S_P$ is isomorphic to $k_A{}^P$ for some $A\subset P$ which is conjugate to $Q$.*

Proof. Since $Y$ is completely reducible, $(Y, S)^G=(Y, S)^G_Q$ and $(Y, S)^G_R=0$ if $R$ does not contain any conjugate of $Q$. Let $X$ be an indecomposable direct summand of $S_P$. Then by Mackey decomposition theorem $X\simeq k_A{}^P$ for some $A\subset P$ such that $A$ is contained in some conjugate of $Q$. By the isomorphism