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Osaka J. Math.
23 (1986), 457–459

ON MODULES THAT COMPLEMENT DIRECT SUMMANDS

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(Received October 5, 1984)

A module M is said to *complement direct summands* if every direct summand of M has the exchange property with respect to completely indecomposable modules, or in other words if for each direct summand B of M and for each decomposition $M=\bigoplus_I A_i$, where every A_i is completely indecomposable (i.e. has local endomorphism ring), there exists a subset K of I with $M=B\oplus\bigoplus_K A_k$.

There are several characterisations by a theorem of Harada [3, 3.1.2].

Theorem. *Let $M=\bigoplus_I A_i$ be a c. indec. decomposition. Equivalent are*

(1) *M satisfies the take-out property.*
(2) *Every direct summand of M has the exchange property in M.*
(3) *M complements direct summands.*
(4) *$(A_i\colon I)$ is a locally-semi-T-nilpotent family.*
(5) *$J'\cap End(M)$ is equal to the Jacobson radical of $End(M)$.*

One step of the proof, "(4)$\Rightarrow$(5)", does merit a certain attention. In an earlier version of the theorem by Harada and Sai [2, Thm 9], the proof of that step uses assumptions stronger than at hand [2, Lemma 12]. We would like to present an alternative and elementary proof of that step. In particular one does not need transfinite induction as in [3, Lemma 2.2.3]. All notation may be found in [3]. For the proofs let perpetually be $M=\bigoplus_I A_i$ a completely indec. decomposition and let $(e_i\colon I)$ be a related set of orthogonal idempotents (i.e. $e_i(M)=A_i$).

By definition, for an element f of End(M) not contained in J', there exist some elements $i, j\in I$ and $g\in$End(M) with $ge_jfe_i=e_i$. Thus the Jacobson radical of End(M) is always contained in $J'\cap$End(M), otherwise it would contain a nonzero idempotent.

Lemma 1. *For all $t\in J'\cap End(M)$ and for all $i\in I$, e_it and te_i are elements of the Jacobson radical.*