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HARMONIC MAPS FROM S^2 TO HP^2

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In this paper, we describe all harmonic maps from the Riemann 2-sphere to $\boldsymbol{H}P^2$, the quaternion 2-projective space. In example 1.3, all isotropic harmonic maps from S^2 to $\boldsymbol{H}P^n$ are given. A particular class of nonisotropic harmonic maps from S^2 to $\boldsymbol{H}P^n$ are classified by Theorem 1.4. With theorem 1.5, the description of harmonic maps from S^2 to $\boldsymbol{H}P^2$ becomes complete.

Harmonic maps from S^2 to S^n and S^2 to CP^n are classified by Calabi. E [1] and Eells-Wood [2] respectively. Our description of harmonic maps from S^2 to $\boldsymbol{H}P^2$ is not as elegant as those of Calabi and Eells-Wood. Still it gives hope for classifying harmonic maps from S^2 to compact symmetric spaces.

We state our main results in §1. §2 contains some preliminaries. In §3, the proof of theorem 1.4 is given. Theorem 1.5 is proved in §4. Here we use some of the ideas from [6].

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1. Main results

$\boldsymbol{H}^n$ denotes the quaternionic space of dimension n over $\boldsymbol{H}$, the quaternions. We have the quaternion metric $\langle\ ,\ \rangle$ on $\boldsymbol{H}^n$ defined $\langle v, w\rangle=\sum a_i \bar{b}_i$ where $v=(a_1, \cdots, a_n)$, $w=(b_1, \cdots, b_n)\in\boldsymbol{H}^n$. For $a\in\boldsymbol{H}$, $\bar{a}$ denotes the conjugation of a in $\boldsymbol{H}$. Write

$$\langle v, w\rangle = H(v, w)+A(v, w)j \tag{1.1}$$

where $H(v, w)$, $A(v, w)\in C=\boldsymbol{R}+\boldsymbol{R}i$. Define T: $\boldsymbol{H}^n\to C^{2n}$ by $T(x_1+y_1j, \cdots, x_n+y_nj)=(x_1, y_1, \cdots x_n, y_n)$. T is a C-linear isomorphism of $\boldsymbol{H}^n$ with C^{2n}. Always, we identify C^{2n} and $\boldsymbol{H}^n$ through this isomorphism. Then H defined in (1.1) is the standard Hermitian metric on C^{2n} and A defined in (1.1) is a nondegenerate alternating C-bilinear form on C^{2n}. Let J denote left multiplication by j in $\boldsymbol{H}^n$. Then $H(v,Jw)=A(v,w)$ and $A(v,Jw)=-H(v,w)$ for $v,w\in\boldsymbol{H}^n$.

For a subspace W of C^{2n}, put

$$W^{\perp} = \{x\in C^{2n}: H(x, y) = 0 \quad \text{for all } y\in W\} \text{ and}$$
$$W_A^{\perp} = \{x\in C^{2n}: A(x, y) = 0 \quad \text{for all } y\in W\}\ .$$