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Osaka J. Math.
23 (1986), 229–247

THE DUALITY CONJECTURE IN FORMAL KNOT THEORY

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(Received December 3, 1984)

0. Introduction

This paper deals with the formal knot theory of Kauffman [2]. In the first three sections we prove his Duality Conjecture ([2], p. 57); in the fourth we give a new and simpler proof of Kauffman's key combinatorial result, his Clock Theorem ([2], Theorem 2.5). This is based on a reformulation of several of the notions of formal knot theory, and in these terms the theorem can be expressed as follows: the set of maximal trees in a connected plane graph has a naturally defined partial order, which gives it the structure of a distributive lattice.

We assume that the reader is familiar with the basic definitions of [2] (universe, state, black and white holes...), most of which are to be found in the introduction. One piece of this terminology seems worthy of explicit mention here: the word "knot" means a universe with a specification of over- and under-crossings; in standard terminology this would be a knot (or link) diagram. Also, we make a minor departure from Kauffman's usage in that we regard a universe as a subset of S^2 rather than $\boldsymbol{R}^2$.

We now outline the proof of the Duality Conjecture. A polynomial $f(B, W)\in \boldsymbol{Z}[B, W]$ will be called *good* if $f(-W, -B)=f(B, W)$. (Note that since $f(B, W)\mapsto f(-W, -B)$ is an automorphism, the good polynomials form a subring of $\boldsymbol{Z}[B, W]$.) It is easy to see that the Duality Conjecture asserts that the state polynomial $\langle U\rangle$ of any universe is good. In fact we prove that $\langle K\rangle$ is good for any knot K. We start from the observation that if we try to prove this by induction on the number of crossings of K, we may change crossings at will. This is because, for knots K and $\bar{K}$ differing at a single crossing, the exchange identity ([2], Theorem 4.1) expresses $\langle K\rangle-\langle\bar{K}\rangle$ as the state polynomial of a knot with fewer crossings. In § 1 we show that a similar situation obtains for knots related by a Reidemeister move. Then in § 2 we show, in effect, that one can use Reidemeister moves and crossing changes to reduce the number of crossings of a knot without ever increasing this number. These

*) Partially supported by NSF Grant MCS-8102118.