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ON COMMUTATORS IN REAL SEMISIMPLE LIE GROUPS

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1. Introduction

Every element of the commutator subgroup G' of a group G is a finite product of commutators. We shall say that G has property (C) if every element of G' is a commutator in G. For finite groups which do not have property (C) see [11], [13, p. 258], and [14].

Ore [19] raised the question whether all finite non-abelian simple groups G have property (C). If the character table of G is known then it is easy to check which conjugacy classes of G consist of commutators, see [14]. Ito [15] showed that the alternating groups A_n $(n\geq 5)$ have property (C), see also [19]. For other results on Ore's question consult [11] and the references therein.

Shoda [22] showed that if k is an algebraically closed field then the group $SL_n(k)$ has property (C). From the results of Thompson [23] it follows that $SL_n(k)$ has property (C) if k is an arbitrary field except in the case when $n\equiv 2$ (mod 4), k contains a primitive n-th root of unity, and the equation $x^2+y^2=-1$ has no solution in k. In the exceptional case only the central elements of order n are not commutators.

Goto [10] showed that connected compact topological group G whose commutator subgroup is dense in G has property (C). In particular, all connected compact semisimple Lie groups have property (C), see also [2, p. 33]. Pasiencier and Wang [20] proved the same result for connected complex semisimple Lie groups. Their result has been extended to connected semisimple algebraic groups over algebraically closed fields by Ree [21].

Note that $-1\in SL_2(\boldsymbol{R})$ is not a commutator of $SL_2(\boldsymbol{R})$ by Thompson's results. More generally, we show in Proposition 1 that if $\varepsilon=\exp(2\pi i/n)$, $n=p+q$, $p\geq q\geq 1$, then $\varepsilon\in SU(p,q)$ is not a commutator in $SU(p,q)$. This includes the previous case because $SU(1,1)\cong SL_2(\boldsymbol{R})$. Thus there exist connected almost simple real Lie groups which do not possess property (C).

Isaacs [14] observes that no example seems to be known of a non-abelian simple group which possesses a non-commutator. We think that the simple

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