Furukawa, T.
Osaka J. Math.
23 (1986), 217–221

THE GROUP OF NORMALIZED UNITS OF A GROUP RING

TÔRU FURUKAWA

(Received December 17, 1984)

Introduction

Let RG be the group ring of a group G over a commutative ring R with identity and $\Delta_R(G)$ its augmentation ideal. For a normal subgroup N of G, the kernel of the natural homomorphism $RG \to R(G/N)$ will be denoted by $\Delta_R(G, N)$. It is equal to $\Delta_R(N)RG$. Also, we shall denote by $V(RG)$ the group of normalized units of RG, that is, $V(RG)=U(RG)\cap(1+\Delta_R(G))$ where $U(RG)$ is the unit group of RG.

The aim of this paper is to prove the following theorem which generalizes [1, Proposition 1.3].

Theorem 1.3. *Let G be an arbitrary group and R an integral domain of characteristic* 0. *Let I be an ideal of RG and set $J=\bigcap_{n=1}^{\infty}(I+\Delta_R(G)^n)$. Then the factor group*

$$V(RG)\cap(1+J)/V(RG)\cap(1+\Delta_R(G, G\cap(1+J)))$$

is torsion-free.

As an immediate consequence of this result we can weaken the condition on R in [1, Proposition 2.4]. To be more precise, let $D_{n,R}(G)$ be the n-th dimension subgroup of G over R. Then, for two groups G and H with isomorphic group algebras over an integral domain R of characteristic 0, we can show that $D_{n,R}(G)=\{1\}$ if and only if $D_{n,R}(H)=\{1\}$.

Let A be a ring and ${}^0A$ the group of all quasi-regular elements in A. Here we say that A is residually nilpotent if $\bigcap_{n=1}^{\infty}A^n=0$. As another application of Theorem 1.3, we show that if A is a residually nilpotent algebra over an integral domain R of characteristic 0, then the group $G={}^0A$ has a torsion-free normal complement in $V(RG)$. This is proved by D.S. Passman and P.F. Smith [3, Theorem 1.4] for the case where A is a finite nilpotent ring and R is the ring of rational integers.

1. The group of normalized units

We start by making two simple observations. Let G be a group, R a