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GENERALIZATIONS OF NAKAYAMA RING I

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T. Nakayama found a very important ring in ring theory, which we call a generalized uniserial ring [6]. He showed that a left and right artinian ring R is a generalized uniserial ring if and only if every (finitely generated) left (resp. right) R-module is a direct sum of uniserial modules. We shall generalize further such a ring from this point of view.

We shall define Conditions $(*, 3)$ and $(**, 3)$ (see §1) for a direct sum $D(3)$ of three hollow modules. If R is a generalized uniserial ring, $(*, 3)$ and $(**, 3)$ are satisfied ([2] and [3]). In §2 we shall give a characterization of a right artinian ring R which satisfies $(**, 3)$ for any $D(3)$. In §3 we shall expose several examples related to the results in the previous section.

We shall study Condition $(*, 3)$ in a forthcoming paper.

1. Definitions. Let R be throughout a right artinian ring with identity. Modules in this note are unitary right R-modules with finite length. Let e be a primitive idempotent in R. If $eR\supset eJ\supset eJ^2\supset\cdots\supset eJ^n=0$ is a unique chain of the submodules of eR for each e, R is called a *right* (*generalized uni-*) *serial ring* (*Nakayama ring*), where $J=\mathrm{J}(R)$ is the Jacobson radical of R.

As a generalization of a serial ring, we have considered the following two conditions [1]:

$(*, n)$ *Every* (*non-zero*) *maximal submodule of a direct sum* $D(n)$ *of* n *non-zero hollow modules is also a direct sum of hollow modules,* and

$(**, n)$ *Every* (*non-zero*) *maximal submodule of the* $D(n)$ *above contains a non-trivial direct summand of* $D(n)$.

By Nakayama [6], if R is a right and left serial ring, $(*, n)$ holds for any $D(n)$ and any n as right (resp. left) R-modules. Further R is a right serial ring if and only if (*, n), replaced hollow by uniserial, holds for any $D(n)$ and any n as right R-modules [5].

In general, if $J^2=0$, $(*, 2)$ holds for any $D(2)$ by [3], Proposition 3. Let $\{N_i\}_{i=1}^n$ be a set of hollow modules, and put $D(n)=\sum_{i=1}^{n}\oplus N_i$. Let M_1 be a maximal submodule of $D(n-1)=\sum_{i=1}^{m-1}\oplus N_i$. Then $M=M_1\oplus N_n$ is a maximal submodule of D. If D satisfies $(*, n)$, M_1 is also a direct sum of hollow modules