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A TRIPLING ON THE ALGEBRAIC NUMBER FIELD

Dedicated to Professor Nagayoshi Iwahori on his 60th birthday

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Throughout this paper we fix an (even or odd) prime number l, $n=l^\nu$ a power of l, and an algebraic number field k which contains $\exp(2\pi i/n)$ and has a finite degree over the rational number field $\boldsymbol{Q}$. For elements x, $y\in k^\times$ and a prime spot $\mathfrak{p}$ in k (abbrev. k-prime), Hilbert's n-th power residue symbol $\left(\frac{x, y\,|\,k}{\mathfrak{p}}\right)_n$ is defined by $\sqrt[n]{y}^{\sigma}=\left(\frac{x, y\,|\,k}{\mathfrak{p}}\right)_n\cdot\sqrt[n]{y}$ using norm residue symbol $\sigma=\left(\frac{x, k(\sqrt[n]{y})/k}{\mathfrak{p}}\right)$, which takes the value in the group of the n-th roots of 1 in k, W. We denote the completion of k at $\mathfrak{p}$ by $k_\mathfrak{p}$ and the group of the n-th roots of 1 in $k_\mathfrak{p}$ by $W_\mathfrak{p}$. After the canonical inclusions of fields $k\subset k_\mathfrak{p}$ and of Galois groups $G(k_\mathfrak{p}(\sqrt[n]{y})/k_\mathfrak{p})\subset G(k(\sqrt[n]{y})/k)$, the (global) Hilbert's power residue symbol is extended to the local symbol $(x, y\,|\,k_\mathfrak{p})_n$ taking the value in $W_\mathfrak{p}$. We put

$$(x, y)_n = \prod\nolimits_{\text{all } k\text{-prime } \mathfrak{p}} (x, y\,|\,k_\mathfrak{p})_n$$

which is a pairing on $k^\times$ taking the value in the group $\prod_\mathfrak{p} W_\mathfrak{p}$. Then this pairing symbol admits fundamental properties like the multiplicativity about each component, the conjugacy theorem about isomorphism $\tau\colon k\to k^\tau$, and the transgression theorem about the lifting of k to some extension k'/k. Moreover it has the norm theorem saying that x is in the norm group $N_{K/k}K^\times$; $K=k(\sqrt[n]{y})$, if and only if $(x, y)_n=1$ and further it has the reciprocity law saying that $(x, y)_n=(y, x)_n^{-1}$.

In this paper we define a tripling symbol $(x, y, z)_n\in W$ on $k^\times$. Not that it is defined for all the elements of $k^\times\times k^\times\times k^\times$, but it is done only for the ones having a property named *strictly orthogonal*. The definition of strict orthogonality is rather complicated but we shall illustrate here some sufficient conditions for it. In case $l\neq 2$, $\{x, y, z\}$ are strictly orthogonal if they are all n-th powers in $k_\mathfrak{l}^\times$ at any $\mathfrak{l}\,|\,(l)$ under inclusion $k\subset k_\mathfrak{l}$ and any two of them are orthogonal about the symbol $(\ ,\ )_n$. In case $l=2$ we need some additional conditions; saying when $\exp(2\pi i/4n)\in k$, $\{x, y, z\}$ are strictly orthogonal if further any two of them are orthogonal about $(\ ,\ )_{2n}$. Our results of this paper are the